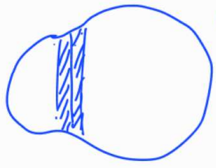
Integration on Riemann Surfaces



Curves / Surfaces
↓ ↓
Areas Volumes

- **Holomorphic 1-forms:**

Def^n: A holomorphic 1-form on an open set $V \subset \mathbb{C}$ is an expression ω of the form
$$\omega = f(z)\,dz$$
where f is a holomorphic function on V.

We say that ω is a holomorphic 1-form in the coordinate z.

Def^n: Suppose that $\omega_1 = f(z)\,dz$ is a holomorphic 1-form in coordinate z (ω) defined on an open set V_1 (resp. V_2) $(\omega_2 = g(\omega)\,d\omega)$

Let $z = T(\omega)$ define a holomorphic mapping from U_2 to V_1

We say that ω_1 transforms to ω_2 under T if $g(\omega) = f(T(\omega))\,T'(\omega)$

Mot^n: [because of $dz = T'(\omega)\,d\omega$] cooked up the above def^n

Note that if T is invertible with the inverse function S, then ω_1 transforms to ω_2 under T iff ω_2 transforms to ω_1 under S.

Def^n: Let X be a R.S. A holomorphic 1-form on X is a collection of holomorphic 1-form $\{\omega_\phi\}$, one for each chart $\phi: U \to V$ in the coordinate of the target V, such that if two charts $\phi_i : U_i \to V_i$ (for $i = 1, 2$) have overlapping domains, then the associated holomorphic 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

Lemma: Let X be a R.S. and A be a complex atlas on X. Suppose

Lemma: Let X be a R.S, and A be a complex atlas on X. Suppose that holomorphic 1-forms are given for each chart of A, which transform to each other on their common domains. Then there exists a unique holomorphic 1-form on X extending these holomorphic 1-forms on each of the charts of A.

Pf: Let ψ be a chart of A.

w: local coordinate of ψ.

$p \in \mathrm{dom}(\psi)$.

choose a chart $\phi \in A$ containing p

local coordinate z. Let $f(z)dz$ be the the holomorphic 1-form wrt ϕ

ψ : $f(T(w))\, T'(w)\, dw$ where $z = T(w)$ describes the change of coordinates $\phi \circ \psi^{-1}$.

- **Check:** · Defn is independent of the choice of ϕ
  · gives a 1-form wrt ψ at every point of the domain
  · All these holomorphic one forms transform into each other and thus define a holomorphic 1-form on X.
  · This 1-form is unique. $\square$.

Ex: On $\mathbb{C}_\infty$, there are no non-zero holomorphic one forms!

Meromorphic 1-forms:

Defn: A meromorphic 1-form on an open set $V \subset \mathbb{C}$ is an expression of the form $\omega = f(z)dz$ where f is a meromorphic function on V.

. We say that ω is a meromorphic one form in the z-coordinate.

Defn: Suppose $\omega_1 = f(z)dz$ (resp. $\omega_2 = g(w)dw$) meromorphic 1-forms coordinates z (resp. w) defined V_1 (resp. V_2)

$z = T(w)$ holomorphic: $V_2 \to V_1$

We say ω_1 transforms to ω_2 under T if

$$g(w) = f(T(w))\, T'(w).$$

Defn: Let X be a R.S. A meromorphic 1-form on X is a collection of meromorphic 1-forms $\{\omega_\phi\}$, one for each chart $\phi: U \to V$ in the variable of the target V, such that if 2 charts overlap, then the associated meromorphic 1-forms ω_{ϕ_1} transfers to ω_{ϕ_2} under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

Lemma: It is good enough to define a meromorphic 1-form on a complex atlas.

Let ω be a meromorphic 1-form defined in a nbhd of a point p. Choosing a local coordinate centered at p; we may write $\omega = f(z)dz$ where f is a meromorphic function at $z=0$.

Defn: The order of ω at p, denoted by $\mathrm{ord}_p(\omega)$ is the order of the function f at 0.

- check: · $\mathrm{ord}_p(\omega)$ is well defined, independent of the choice of local coordinate.
  - A meromorphic 1-form ω is holomorphic at p iff $\mathrm{ord}_p(\omega) \geq 0$
  - We say p is a zero of ω of order n if $\mathrm{ord}_p(\omega) = n > 0$.
  - We say p is a pole of order n if $\mathrm{ord}_p(\omega) = -n < 0$.
  - The set of zeroes & poles of a meromorphic 1-form is a discrete set.

- If we know that a "global" 1-form exists, then it is good enough to just describe it on a single chart using a single formula. (Identity thm for meromorphic f^n).
- But if we don't know this, and we start by defining a ... $f(z)\,dz$ in one chart, then

meromorphic local expression ... to another when one transforms this local expression to another chart it may fail to be meromorphic.

For example, the meromorphic 1-form $\exp(z)dz$ on the finite chart $\mathbb{C}$ of $\mathbb{C}_\infty$ does not extend to a meromorphic 1-form in a nbhd of ∞.

Ex:

. There is also another problem: local expression does not transform uniquely to the other points of X.

Eg: $\sqrt{z}\,dz$ mero 1-form $\mathbb{C} \setminus \{\mathbb{R}_-\}$

where the branch of the square root is choosen so that $\sqrt{1} = 1$.

This can be extended to the negative real axis but not uniquely. Hence we do not obtain a meromorphic 1-form on all of $\mathbb{C}^\times$.

Examples: $\mathbb{C}_\infty$: Meromorphic one forms.

then in z-chart: $\omega = f(z)\,dz$ where f is a rational function

($1/z$ chart?)

$1/z\, dz$.

$\mathbb{C}/L$: dz in every chart is a well defined holomorphic 1-form!

C^∞. 1-forms.

locally the expression: $f(x,y)\,dx + g(x,y)\,dy$, where x & y are the local real variables (i.e. $z = x+iy$)

We would rather prefer to use z & $\bar{z}$
dz & $d\bar{z}$.

because $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/2i$

$dx = (dz + d\bar{z})/2$ & $dy = (dz - d\bar{z})/2i$

$f(x,y)\,dx + g(x,y)\,dy \rightsquigarrow r(z,\bar{z})\,dz + s(z,\bar{z})\,d\bar{z}$.

Also do this for partial derivaties

$$\partial/\partial z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

C^∞ function f is holomorphic on an open set U iff $\frac{\partial f}{\partial \bar{z}} = 0$.

C^∞ 1-form : A C^∞ 1-form on an open set $U \subset \mathbb{C}$ is an expression ω of the form

$$\omega = f(z,\bar{z})\,dz + g(z,\bar{z})\,d\bar{z}$$

where f and g are C^∞ function on U. We say that ω is a C^∞ 1-form in the coordinate z.

Transformation rules : $\omega_1 = f_1(z,\bar{z})\,dz + g_1(z,\bar{z})\,d\bar{z}$ $\quad V_1$

$\omega_2 = f_2(w,\bar{w})\,dw + g(w,\bar{w})\,d\bar{w}$ $\quad V_2$, $\quad V_2 \xrightarrow{T} V_1$

$z = T(w)$.

We say that ω_1 transforms to ω_2 under T if

$$f_2(w,\bar{w}) = f_1(T(w), \overline{T(w)})\,T'(w)$$

$g_2(w,\bar{w}) = g_1(T(w), \overline{T(w)})\, T'(w)$

[Reason we use $z, \bar{z}$ is to avoid mixing of coordinates

$dz \to dw$

$d\bar{z} \to d\bar{w}$] .

Transporting to R.S :

Defn: Let X be a R.S. A C^∞ 1-form on X is a collection of C^∞ 1-forms $\{\omega_\phi\}$, one for each chart $\phi : U \to V$ in the variable of the target V, such that if two charts $\phi_i : U_i \to V_i$ (for each $i = 1, 2$) have overlapping domains then the associated C^∞ 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

Lemma: It is is good enough to define on a complex atlas.

1-form of type $(1,0)$ & $(0,1)$:

Def: A C^∞ 1-form is of type $(1,0)$ if it is locally of the form $f(z,\bar{z})\,dz$. It is of type $(0,1)$ if it is locally of the form $g(z,\bar{z})\,d\bar{z}$.

- Any holomorphic 1-form is of type $(1,0)$.

2-forms

Defn: A C^∞ 2-form on an open set $V \subset \mathbb{C}$ is an expression of the form
$$\eta = f(z,\bar{z})\, dz \wedge d\bar{z}$$
$dz = dx + i\,dy$

$d\bar{z} = dx - i\,dy$

where f is a C^∞ function on V. We say that η is a C^∞ 2-form in the coordinate z.

[wedge product?]

- $dz \wedge d\bar{z} = -d\bar{z} \wedge dz$.

$. \; dz \wedge dz = d\bar{z} \wedge d\bar{z} = 0 .$

Transformation rule : $\eta_1 = f(z,\bar{z})\, dz \wedge d\bar{z}$

$$\begin{array}{c} U_1 \quad z \\ \uparrow T \\ V_2 \quad \omega . \end{array}$$

$\eta_2 = g(\omega,\bar{\omega})\, d\omega \wedge d\bar{\omega}$

$$g(\omega,\bar{\omega}) = f(T(\omega), \overline{T(\omega)}) \, \| T'(\omega) \|^2 .$$

$[$ because $\| T'(\omega) \|^2 = T'(\omega) \overline{T'(\omega)}]$.

$dz \to T'(\omega) d\omega$

$d\bar{z} \to \overline{T}'(\omega) d\bar{\omega}$

Transport to R.S :

Def^n: Let X be a R.S . A C^∞ 2-form on X is a collection of C^∞ 2-form $\{\eta_\phi\}$, one for each chart $\phi : U \to V$ in the variable of the target V, such that if two charts $\phi_i : U_i \to V_i$ (for $i = 1, 2$) have overlapping domains, then the associated C^∞ 2-form η_{ϕ_1} transforms to η_{ϕ_2} under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

Lemma: Enough to describe on a complex atlas. $\square$