Deadline for Ex. sheet submission – 27^{th} Dec 2021

Final Exam will be uploaded after the quiz – tomorrow
submission
Deadline – 27^{th} Dec.

Integration of 2-forms:

Let T be a triangle on a R.S. X. Suppose that T is contained completely inside the domain of a chart $\phi : U \to V$. Then if η is a C^∞ 2-form on X, write $\eta = f(z,\bar{z})\, dz \wedge d\bar{z}$

$$\iint_T \eta := \iint_{\phi(T)} f(z,\bar{z})\, dz \wedge d\bar{z}$$

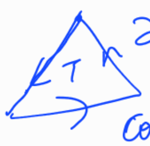
$$= \iint_{\phi(T)} (-2i)\, f(x+iy,\, x-iy)\, dx \wedge dy$$

where this last integral is the usual surface integral in $\mathbb{C} = \mathbb{R}^2$.

Lemma: If $D \subseteq X$ is a triangulable closed set. Then defining

$$\iint_D \eta = \sum_{\{T_\Delta\}_{\Delta \in D}} \iint_T \eta \qquad \eta \ C^\infty \text{ 2-form on } X.$$

is well defined.



$\partial T \subset X$
completely contained in a chart

∂T is a closed path on X

$D \subset X$ triangulable, closed set.

$\{T_i\}$ triangulation

$\partial D = \sum \partial T_i$ chains on X.

called <u>boundary</u> <u>chain</u> of D

<u>Stokes' Theorem</u> :

<u>Theorem</u>: let D be a triangulable closed set on a R.S X and let ω be a C^∞ 1-form on X. Then

$$\int_{\partial D} \omega = \iint_D d\omega .$$

<u>Residue theorem</u> : let ω be a meromorphic 1-form on a compact R.S. X. Then

$$\sum_{p \in X} \operatorname{Res}_p(\omega) = 0 .$$

<u>Proof</u>: let $p_1, \dots, p_n$ be poles of ω (they form a finite set).

γ_i be a small path on X enclosing p_i and no other pole of ω, and let U_i be the interior of γ_i.

Residue thm in Complex plane

$$\int_{\gamma_i} \omega = 2\pi i \operatorname{Res}_{p_i}(\omega) .$$

[lemma before from yesterday].

Let <u>$D = X \setminus \bigcup_i U_i$</u> $\therefore$ then D is triangulable

$$\partial D = -\sum_i \gamma_i \text{ as a chain on } X .$$

Therefore:

$$\sum_i \operatorname{Res}_{p_i}(\omega) = \frac{1}{2\pi i} \sum_i \int_{\gamma_i} \omega$$

$$= \frac{-1}{2\pi i} \int_{-\sum \gamma_i} \omega$$

$$= \frac{-1}{2\pi i} \iint_{\partial D} \omega$$

$$= \frac{-1}{} \iint d\omega \quad \text{Stokes's thm.}$$

$2\pi i \iint_D$

$= 0 \qquad 0 .$ [ω is holomorphic]

$\square$

The Residue thm is used in the proof of Riemann-Roch theorem: Describes precisely the space of meromorphic functions with prescribed poles on a compact R.S.

Homotopy & Integration & Period map:

Let $\Gamma : [a,b] \times [0,1] \longrightarrow X$ be a cont^n function.

For each $s \in [0,1]$, define $\gamma_s : [a,b] \to X$ paths on X

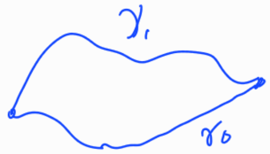
$$\gamma_s(t) = \Gamma(t,s)$$

Assume all γ_s have same end points; i.e, the map Γ is constant on the two sets $\{a\} \times [0,1]$ and $\{b\} \times [0,1]$.

Def^n: A map Γ as above defines a homotopy b/w the paths γ_0 and γ_1 on X.

We say the two paths γ_0 & γ_1 are homotopic via Γ.

Prop^n: Suppose γ_0 & γ_1 are homotopic paths on a R.S X. Then if ω is any closed 1-form on X (i.e $d\omega = 0$) then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.



Pf: D is the image of the rectangle under homotopy, then D is triangulal and $\partial D = \gamma_1 - \gamma_0$

$$\int \omega = \int \omega = \iint_D d\omega = 0$$

$$\int_{\gamma_1 - \gamma_0} \qquad \int_{\partial D} \qquad D \downarrow 0 .$$

$\square$

In particular for any holomorphic 1-form, the integral depends only on the homotopy class of the path of integration not on the path itself.

Let $\pi_1(X, p)$ be the fundamental gp. based at $p \in X$

For any closed 1-form ω

$$\int_{-} \omega : \pi_1(X, p) \longrightarrow \mathbb{C} \quad \leftarrow \text{abelian group}$$

$$[\gamma] \longmapsto \int_\gamma \omega$$

group morphism factors via abellanization the fundamental group.

$$\int_{-} \omega : \underbrace{\pi_1(X, p) / [\pi_1, \pi_1]}_{\substack{\| \\ H_1(X, \mathbb{Z})}} \longrightarrow \mathbb{C}. \quad (\text{period map for 1-form } \omega).$$

$H_1(X, \mathbb{Z})$ first homology group of X.

[X R.S. genus g, $H_1(X)$ = free abelian group of rank $2g$].

$\square$

" ~~Weil~~ Weil. Cohomology theory " —.