Recall: • Nodes
• Compact Riemann surfaces constructed by resolving nodes.

Prop^n: Let $F(x,y,z)$ be an irreducible polynomial of degree d, defining the locus of roots $X \subset \mathbb{P}^2$. Assume that at all but finitely many points of X, F is a non-singular polynomial, that is, at least one of its first partial derivatives is non-zero.
Assume further that these finitely many singular pts are nodes of X.
Then the compact R.S. obtained by resolving these nodes X is a compact R.S.

Genus of a projective Plane curve with nodes:

Prop^n: (Plücker Formula) Let X be a projective plane curve of degree d with n nodes and no other singularities. Then the genus g of X is

$$g = \frac{(d-1)(d-2)}{2} - n.$$

Proof: (We will do a bit later).

Thm: If a Riemann surface of genus g is not hyperelliptic, then it it can be obtained as a projective plane curve of degree $2g-2$ with exactly $2g^2 - 8g + 6$ nodes!

Resolving Monomial Singularities:

Def^n: The singularity of $f(x,y)=0$ at the origin is said to be (n,m)-monomial if there are power series $g(x,y)$ and $h(x,y)$, each having no constant term, and having linearly independent linear terms, such that $f(x,y) = g(x,y)^n - h(x,y)^m$ as power series in two variables.

Notes: if either n or $m = 1$, then the curve is non-singular at origin!
$\Rightarrow$ w.l.o.g. we may assume both n & $m \geq 2$.

- Node is just a $(2,2)$ monomial singularity

[Node to look locally like $xy=0$, set $g=(x+y)/2$, $h=(x-y)/2$, $xy = g^2-h^2$.]

- The existence of the power series g and h give us a second pair of local analytic coordinates $z=g(x,y)$ & $w=h(x,y)$ on the plane

$\Rightarrow$ we can rewrite the defining eq^n as $z^n = w^m$

(This is going to be our prototype for (n,m)-monomial singularity).

X (z,w)

- Assuming that n, m are relatively prime

$\Rightarrow$ choose integers a, b such that $an+bm=1$

$\leftsquigarrow$ hole chart.

$$u: \mathbb{C} \longrightarrow X \qquad s: X \longrightarrow \mathbb{C}$$
$$t \longmapsto (t^m, t^n) \qquad (z,w) \longmapsto t = z^b w^a$$
$$(z'', w)$$

Ex: u and s are inverse maps between a neighborhood of $t=0$ and a neighborhood of the monomial singularity on the singular curve.

So, s gives us a hole chart on the curve with the singular point deleted. And plugging this hole resolves the singularity.

- Assume that $n=m$. If we let $\zeta = \exp(2\pi i/n)$ nth-primitive root of unity, then

$$z^n - w^n = \prod_{i=0}^{n-1} (z - \zeta^i w)$$

Each of these factors define a smooth curve! and all of them pass through the origin, which is of course the singular point!

- Remove the origin
- Gives a space which decomposes into n smooth curves. each with a hole in it.
- Plugging these n holes resolves the singularity in this case.

- Assume that $n < m$ and $(n,m) = k$ with $1 < k \leq n$

The resolution of this singularity simply combines the previous two cases:

Write $n = k\alpha$ $\quad m = k\beta$, then $(\alpha, \beta) = 1$

$\zeta = (2\pi i/k)$ primitive k^{th} root of unity.

$$z^n - w^m = (z^\alpha)^k - (z^\beta)^k = \prod_{i=0}^{k-1} (z^\alpha - \zeta^i w^\beta)$$

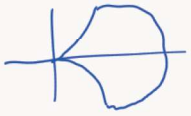
Each of the factor we know how to deal with.

Lemma: A plane curve singularity which is (n,m)- monomial is resolved by the above process, which involves removing the singular pt. & plugging the resulting $k = (n,m)$ holes.

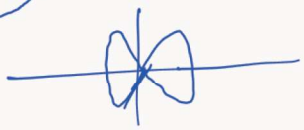
Special names

$(2,2)$ - monomial singularities — Nodes.

$(2,3)$ - Cusp



$(2,4)$ - tacnode



$(2,m)$ - higher order cusps
$m \geq 5$, odd

$(2,m)$ - higher order tacnodes.
$m \geq 6$, even

$(2,m) \longleftrightarrow$ type $a_{m-1} \longrightarrow$ semi simple lie algebra.

[McKay Correspondences)

Advertisment for Lie algebras

- Semisimple Lie algebras.
- Can be classified using dynkin diagram
- Combinatorial data.

A $(3,3)$ - ordinary triple point $\longleftrightarrow$ type d_4

$(3,4)$ — type e_6

$(3,5)$ — type e_8

$(3,6)$ — infinitely near triple point.

(n,n) - monomial singularity ordinary n-fold point

Defn: A plane curve singularity $f(x,y) = 0$ at the origin is an ordinary n-fold point if the lowest term of f is the degree n term and this term (which is a homogeneous polynomial of

degree n in x & y) completely factors into distinct linear factors.

$f = xy + \text{hot}$ $(2,2)$

$f = xy^2 + \text{hot}$.

- The entire polynomial $f(x,y)$ factors completely into n power series, each of which is smooth at the origin.

- Resolve ordinary n-fold point. remove the singular point & plug the n-holes.

Assume $f(x,y) = gh$ polynomial

each g, h curves has monomial singularities only.

Resolve singularities of f by resolving singularties of g & h as curves.

Eg: Singularities of type d_n, $n \geq 4$.

eqn of form $x(y^2 - x^{n-2})$

eg type: $x(x^2 - y^3)$

The singularities of type a_n $(n \geq 1)$, d_n $(n \geq 4)$ and e_n $(n = 6, 7, 8)$ are called simple plane curve singularities.

Cyclic Coverings of the line.

(*) Choose an integer d and a polynomial $h(x)$ of degree k with distinct roots.

Consider the affine plane curve X defined by the eqn

$$y^d = h(x).$$

Since h has distinct roots $\Rightarrow X$ is smooth

Claim: X has finitely many holes at infinity. when resolve these holes we obtain a compact R.S. Y.

Let $x = 1/z$ so that $y^d = h(1/z)$ - (*)

Write $k = dl - \epsilon$, $0 \le \epsilon < d$ $\quad k < d(k-1)$.

Multiply (*) by z^{dl} $\Rightarrow$ $(yz^l)^d = z^{dl} h(1/z) = z^\epsilon (z^k h(1/z))$

Now let $w = yz^l$

$$g(z) = z^k h(1/z).$$

- g is a polynomial with a non-zero constant term c, and near $z=0$ ($x = \infty$), the curve X is described by the eqn $w^d = z^\epsilon g(z)$ with z near 0.
- If $\epsilon = 0$, $z \to 0$, we see that $w \to$ any of the d dth root of c.
- The projection to z gives d hole charts on X.
  near $x = \infty$; plugging these gives a compact R.S. Y.
- If $\epsilon \neq 0$ then as $z \to 0$, $w \to 0$.

  By choosing ϵ^{th} root of $g(z)$, absorb the function g into z^ϵ and note that the new curve we have is nothing but a monomial singularity curve $w^d = z^\epsilon$.

  Resolve the monomial singularity to produce a R.S. remove the singular point (if $\epsilon \ge 2$) and plug the resulting (d, ϵ) holes.

Thus, we have compactified X to a compact R.S. Y by plugging certain holes at $x = \infty$.

- $h(x)$ having distinct roots is unnecessary assumption.
  if h has multiple root at $x = x_0$, then one simply gets a monomial singularity and we know how to resolve it.

Any R.S. obtained in this way, by resolving and compactifying a plane curve defined by $y^d = h(x)$ is called a

cyclic covering of the line.

- Natural projection $\pi : Y \to \mathbb{P}^1$
  $(x,y) \longmapsto x$.
  holomorphic map.

"cyclicity" : $\exists$ an automorphism σ of Y generalizing the hyperelliptic involution. Choose a d^{th} root of unity ζ, and consider the map $\sigma : Y \to Y$
$(x,y) \mapsto (x, \zeta y)$
Order of σ is d.

commutes with π i.e. $\pi \circ \sigma = \pi$. ∎

Methods we know so far on how to compactify plane curves:

1) One point compactification – Complex st^r compatibility with the complex st^r already present on X.

2) Take projective closure. (Projectivization) $\longrightarrow$? do we get a R.S ? (Smoothness is not carried over)

3) Plugging holes – $\begin{cases} \text{nodes} \\ \text{(monomial singularities)} \\ \text{Cyclic covering of lines.} \end{cases}$

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