FINAL EXAM: ALGEBRAIC CURVES AND RIEMANN SURFACES

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We will do a proof of a very important theorem in algebra for which one usually studies a completely different theory (which one?) using Riemann surfaces and Monodromy groups.

The theory of Riemann surfaces was originally developed by Riemann to find a "good way" to think of multivalued complex functions, e.g. $f(z) = \sqrt{z} : \mathbb{C} \to \mathbb{C}$. The idea he had was that can we change the space on which this function lives, so that we can see the multivalued functions as a single-valued functions.

Let us describe in details the case of $w := f(z) = \sqrt{z}$ and look at it as a single valued function: Cut the z-plane $\mathbb{C}$ (the domain of f) along the negative side of the real axis from 0 to $-\infty$, and for every z not belonging to the cut let us choose the value $w = \sqrt{z}$ which lies on the right half w plane (the range of f). So we have made a continuous single-valued function over the whole z plane, except the cut. Let us denote this function as $\sqrt{z}_r$, it defines a continuous and single valued mapping of the z plane, except the cut, on the right half w plane. Similarly, for every z not belonging to the cut we could choose the value $w = \sqrt{z}$ which lies on the left half w plane (the range of f). Let us denote this function as $\sqrt{z}_l$, it defines a continuous and single valued mapping of the z plane, except the cut, on the left half w plane.

(1) (1 pt) Draw the functions $\sqrt{z}_r$, $\sqrt{z}_l$ and show where the two points on either side of the cut are mapped to on the w-plane by these functions.
Here we have $\sqrt{z}_l = -\sqrt{z}_r$.

Functions $\sqrt{z}_l$ and $\sqrt{z}_r$ so defined are called the **continuous single valued branches** of the function $w = \sqrt{z}$. This concept you, most probably, have seen in your complex analysis course. Consider now two copies of the z plane, which we will call **sheets**, and cut every sheet along the negative side of the real axis from 0 to $-\infty$. Take the function $\sqrt{z}_r$ on the first sheet and $\sqrt{z}_l$ on the second sheet. Thus we can see the functions $\sqrt{z}_l$ and $\sqrt{z}_r$ as a unique single valued function, defined no longer on the z plane but on a more complex surface consisting of two distinct sheets. But this is not satisfactory. Indeed, if a point z moves continuously on the first (or on the second) sheet, not crossing the cut, the single-valued function we have defined varies continuously. But if the point z moving on the first sheet, transverses the cut, we lose the continuity!

However, look back at your drawings in above question and note that if traversing the cut, the point z moves from the upper side of one sheet to the lower side of the other sheet, the single valued function we have defined varies continuously. To ensure that the point z moves as desired, we take the upper side of the cut on the first sheet and and join it with the lower side of the second sheet, and the lower side of the cut on the first sheet is joined to the upper side of the cut on the second sheet. Furthermore, when joining the sheets we add between them the real negative axis from the point 0 to $-\infty$.

(2) (1 pt) Draw the surface we have constructed.

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Thus we have transformed the 2-valued function $w = \sqrt{z}$ into another function which is single valued, continuous, no longer on the complex plane but on a new surface. This surface is called the **Riemann Surface of the function** $w = \sqrt{z}$.

(3) (3 pts) Prove that it is indeed a Riemann surface. Which (affine) Riemann surface is it? Compactify it. What is its genus? So, what is it topologically? How do you explain the self intersections on the negative real axis $[0, \infty)$?

The above method can be generalized to construct a Riemann surface of a multi valued algebraic function $w(z)$. Start by separating the single-valued branches of the function $w(z)$, excluding the points z which belong to the cuts. Then we join the branches obtained, choosing the values on the cuts in such a way as to obtain a continuous single valued function on the whole surface. The surface so obtained is called the Riemann surface of the multivalued algebraic function $w(z)$. Let us define precisely what functions we are talking about:

Definition 1. An analytic function $w = f(z)$ or written as just $w(z)$ is called an **algebraic function** if it satisfies a functional equation

$$a_0(z)w^n + a_1(z)w^{n-1} + \ldots + a_n(z) = 0,\ a_0(z) \neq 0,$$

in which the $a_i(z)$ are polynomials in z with complex numbers as coefficients.

From this algebraic equation in w, for $n > 1$, we observe that each value of z determines several values of w, so that w is a multiple valued function of z.

Question: How do we find the single valued branches?

Let $w(z)$ be a multi-valued algebraic function, and fix one of the values of the function $w(z)$ at some point z_0. Let $w_b(z)$ be a continuous single valued branch of the function $w(z)$ defined on some region of the z plane (for example, on the whole plane, except some cuts) and such that $w_b(z_0) = w_0$. Suppose, moreover, that there exists a continuous path C connecting z_0 to a point z_1, lying entirely in the region of the plane considered. Thus, as the point z moves continuously along the path C from z_0 to z_1, the function $w_b(z)$ varies continuously from $w_b(z_0)$ to $w_b(z_1)$.

Let us turn this into a definition!

Suppose at some point z_0 one of the values w_0 of the function $w(z)$ be chosen. Let C be a continuous path beginning at z_0 and ending at certain point z_1. Moving along the path C we choose for every point z on C one of the values of the function $w(z)$ in such a way that these values vary continuously while we move along the path C starting from the value w_0. So, when we arrive at the point z_1, the value of $w(z_1)$ is completely defined.

We say that w_1 is the value of $w(z_1)$ **defined by continuity along the path** C under the condition $w(z_0) = w_0$. Let C_w be the set of the images of all points on the the path C in the w-plane, then it is actually a continuous path in the w-plane beginning at w_0 and ending at w_1, and it is one of the continuous images of the path C under the mapping $w = w(z)$.

There are some evident problems with the above way of defining a function using continuity along a path.

(4) (1 pt) Find all continuous images $w_0(t)$ of a path C with parametric equation $z(t) = 2t - 1$ under the mapping $w = \sqrt{z}$, beginning at a) the point i and b) at point $-i$. Where did you lose the uniqueness?

(5) (1 pt) For $w(z) = \sqrt{z}$, take $w(1) = 1$ and define by continuity $w(-1)$ along the paths: a) upper semi circle of radius 1 centered at origin and b) lower semicircle centered around the origin.

Note again we obtain different values for $w(-1)$. Let us try to find a way to get unique values for $w(-1)$.

(6) (1 pt) Let C be a closed path on the z-plane (i.e., $z(0) = z(1)$). Prove that the value of the function $\sqrt{z}$ at the end point of the path C, defined by continuity, coincides with the value at the initial point iff the path C wraps around the point $z = 0$ an even number of times.

(7) (1 pt) Let C_1 and C_2 be two paths, joining points z_1 and z_2 and let one of the values of $\sqrt{z_0} = w_0$ be chosen. Prove that the values of $\sqrt{z_1}$ defined by continuity along the path C_1 and C_2 are equal if and only if the path $C_1^{-1}C_2$ turns around the point $z = 0$ an even number of times, where C_1^{-1} is the reverse path of C_1 and we are concatenating two paths to get a new path.

From the above statement it follows that if the path $C_1^{-1}C_2$ turns zero times around the point $z = 0$, then the values of the function $\sqrt{z}$ at the final points of the curves C_1 and C_2 will coincide if the values at the initial point coincide. Thus, to separate the single valued branches of the function $\sqrt{z}$, it suffices to take the paths C_1 and C_2 in such a way so that the path $C_1^{-1}C_2$ does not turn around the point $z = 0$. To achieve this, one can make a cut from the point $z = 0$ to infinity to avoid intersecting the path.

(8) (1 pt) Fix the value $w_l = \sqrt{z_0}_l$ at a certain point z_0 and define the values of the function $\sqrt{z}$ at the other points of the z plane (except the cut) by continuity along the curves starting from z_0 and not intersecting the cut. Prove that the continuous single valued branches so obtained coincide with the function $\sqrt{z}_l$ (defined by the value at the point z_0).

Thus, the splitting depends only on the way the cut was made.

(9) (1 pt) Suppose that points z_0 and z_1 do not lie on the cut and that the path C, joining them, transverses the cut once. Choose a value $w_0 = \sqrt{z_0}$ and by continuity along C define the value $w_1 = \sqrt{z_1}$. prove that values w_0 and w_1 correspond to different branches of $\sqrt{z}$.

Thus on transversing the cut, one moves from one branch to the other, i.e., the branches join each other exactly as we have put them together joining the sheets. One obtains in this way the Riemann surface of the function $\sqrt{z}$. This leads to the following definition:

Definition 2. Points, around which one may turn and move from one sheet to another (i.e., changing the value of the function) are called **the branch points** of the given multi-valued function.

(10) (1 pt) So how are these branch points related to the branch points for holomorphic maps of Riemann surfaces (explain for the case of affine curve $w = \sqrt{z}$)?

When a multi-valued function has several branch points, in order to separate the single-valued continuous branches we can make the cuts from every branch point to infinity along lines which do not intersect each other. Another way could be to make cuts (non- intersecting ones) along a line joining two branch points, so if total number of branch points are finite and even, we can do this without involving infinity but if there are odd number of points, make the last branch cut by deleting the line joining the unpaired branch point and infinity.

(11) (2 pts) Find the branch points of the following a) $\sqrt[3]{z^2-1}$ b) $\sqrt{z(z-i)}$ c) $\sqrt{(z^2-1)(z^2-k^2)}$, for $k \neq \pm 1 \in \mathbb{C}$.

(12) (6 pts) Draw their Riemann surfaces, using the branch cuts in the two different ways as described above. Also draw their compactified versions!

(13) (5 pts) Separate the single-valued continuous branches of the function $\sqrt{z^2}$. What is the point $z = 0$ (Yes, it is not a branch point!)? What are the branch points for this function? Use a construction explained in class to construct a Riemann surface for $\sqrt{z^2}$. Compactify it. What is its genus?

The point $z = 0$ is not a branch point of the function $\sqrt{z^2}$. However, the images of the paths passing through the point $z = 0$ are not uniquely defined! Points where the uniqueness of the continuous images of the curves is lost but that are not branch points are called the **non-uniqueness** points of a given function. When building the Riemann surfaces one should draw no cuts from the points of non-uniqueness to infinity or the branch points: in drawing any cuts these points must always be avoided.

We can also consider functions which are defined at some points (the points on which the function is not defined can turn out to be a branch point for the function as well.)

(14) (1 pt) What is the branch point for $\sqrt{1/z}$ on the complex plane?

Let us rephrase our ability to find single valued branches for a multivalued function to monodromy. Suppose a multi-valued function $w(z)$ be such that if one fixes one of its values w_0 at an arbitrary point z_0 then the value of the function can be defined by continuity (possibly in a unique way) along an arbitrary path beginning at the point z_0 (and not passing through the points at which $w(z)$ is not defined). We say that the function $w(z)$ possesses the monodromy property if it satisfies the following condition:

Monodromy Property Suppose that two continuous paths C_1 and C_2 on the z plane join two points z_0 and z_1 passing neither through the branch points nor through the points of non-uniqueness of the function $w(z)$. Furthermore, suppose that the path C_1 can be transformed, varying continuously, into the path C_2 in such a way that none of the path during the deformation passes through the branch points, and that the ends of these paths are fixed. (In other words, the paths C_1 and C_2 are homotopic and the homotopy map does not contain branch points in its image.) Then the value $w(z_1)$ is uniquely defined by continuity along the path C_1 and C_2 (when a value $w_0 = w(z_0)$ is chosen).

(15) (1 pt) Suppose a function $w(z)$ possess the monodromy property. On the z-plane make the cuts, not intersecting each other, from the branch points of $w(z)$ to infinity. Prove that in this way the functions $w(z)$ is decomposed into single valued branches.

(16) (1 pt) In continuation of the setting of previous question, now suppose that the cuts do not pass through the non-uniqueness points of the function $w(z)$ and that $w(z)$ has a finite number of branch points. Prove that on traversing the cut (in a defined direction) one moves from a given branch point of the function $w(z)$ to another, a well defined one, independently of the actual point at which the cut is crossed.

From the above it follows that if a multi valued function $w(z)$ posses the monodromy property, then one can build its Riemann surface. In order to understand the structure of this surface it suffices to find the branch points of the function $w(z)$ and to define the passages between the branches corresponding to the turns around these points.

Question: Which functions have monodromy property?

(17) (1 pt) Suppose that a function satisfies conditions of the monodromy property. Let C_1' and C_2' be the continuous images of the curves C_1 and C_2 under the mapping $w(z)$, with $w_0 = w(z_0)$ as the initial point. Prove that the curves C_1' and C_2' end at the same point.

Fact: All analytic functions satisfy the monodromy property.

Let us change gears and introduce another main player: Functions representable by radicals.

Definition 3. Let $f(z)$ and $g(z)$ be two multi-valued functions. By $f(z) + g(z)$ we will denote the multi-valued function whose values at a point z_0 are obtained by adding each value $f(z_0)$ to each value of $g(z_0)$. Similarly one defines the functions $f(z) - g(z)$, $f(z).g(z)$ and $f(z)/g(z)$.

By $[f(z)]^n$, for $n \in \mathbb{Z}_{>0}$, we denote the function whose values at the point z_0 are obtained raising to power n each value $f(z_0)$.

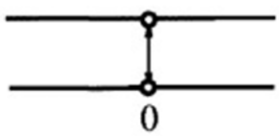
By $\sqrt[n]{f(z)}$, for $n \in \mathbb{Z}_{>0}$, we denote the function whose values at a point z_0 are obtained extracting all roots of order n of each value $f(z_0)$.

Definition 4. We will say that a function $h(z)$ is **representable by radicals** if it can be written in terms of the function $f(z) = z$ and of constant functions ($g(z) = a$, a being any complex number) by means of the operations of addition, subtraction, multiplication, division, raising to an integer power and extraction of a root of integer order.

(18) (4 pts) Let $h(z)$ be a function representable by radicals and let C be a continuous path on the z plane, beginning at a point z_0 and not passing through the points at which $h(z)$ is not defined. Prove that if w_0 is one of the values $h(z_0)$ then there exist at least one continuous image of the path C under the mapping $w = h(z)$, beginning at the point w_0.

From the above, we can conclude that an arbitrary function $h(z)$ representable by radicals can be defined by continuity along an arbitrary continuous path C, not passing through the points at which $h(z)$ is not defined. Moreover, if the path C does not pass through the branch points nor through those of non-uniqueness of the function $h(z)$, then the function $h(z)$ is uniquely defined by continuity along the path C. And we have already seen it (as a fact) that every function representable by radicals satisfies the monodromy property. Hence, using the method from the previous exercises, we can construct a Riemann surface for every function representable by radicals.

Actually the method described above to construct a Riemann surface can be represented pictorially as well in the following way: For each single valued branch sheets in the Riemann surface constructed we draw a line and mark the branch points on the lines. We connect the same branch point on different sheets by arrows for which moving around the branch point, we change sheets. Such pictorial representations are called **Riemann Scheme of an (analytic) function**. The Riemann scheme of $\sqrt{z}$ looks like:



We call the lines as sheets of the Riemann scheme.

(19) (3 pts) Draw the Riemann scheme of the following functions a) $\sqrt[4]{(z-1)^2(z+1)^3}$ b) $\sqrt[3]{z^2-1}$ c) $\sqrt[4]{z^2}$ d) $\sqrt{1/(z-i)}$ e) $\sqrt[3]{(z^2+1)^2}$ f) $\sqrt[4]{(z+i)^2/(z(z-1)^3)}$.

(20) (4 pts) Prove that to build the schemes of the Riemann surfaces of the functions $h(z) = f(z)+g(z)$, $h(z) = f(z)-g(z)$, $h(z) = f(z).g(z)$, $h(z) = f(z)/g(z)$ starting from the schemes of the Riemann surfaces of the functions $f(z)$ and $g(z)$, with the same cuts, it suffices to do the following:
  (a) put into correspondence with every pair of branches, $f_i(z)$ and $g_j(z)$, a sheet on which the branch $h_{i,j}(z)$, equal respectively to $f_i(z)+g_j(z)$, $f_i(z)-g_j(z)$, $f_i(z).g_j(z)$, $f_i(z)/g_j(z)$, is defined;
  (b) if by turning around the point z_0 one moves from the branch $f_{i_1}(z)$ to the branch $f_{i_2}(z)$ and from the branch $g_{j_1}(z)$ to the branch $g_{j_2}(z)$, then for the function $h(z)$ by the same turn one moves from the branch $h_{i_1,j_1}(z)$ to the branch $h_{i_2,j_2}(z)$;
  (c) identify the sheets on which the branches $h_{i,j}$.

(21) (2 pts) Prove that to build the scheme of the Riemann surface of the function $h(z) = [f(z)]^n$ starting from the scheme of the Riemann surface of the function $f(z)$, defined by the same cuts, it suffices to do the following:
  (a) in the scheme of the Riemann surface of the function $f(z)$ consider, instead of the branches $f_i(z)$, the branches $h_i(z) = [f_i(z)]^n$.
  (b) identify the sheets on which the branches $h_i(z)$ coincide.

(22) (4 pts) Prove that to build the scheme of the Riemann surface of the function $h(z) = \sqrt[n]{f(z)}$ starting from the scheme of the Riemann surface of the function $f(z)$, defined by the same cuts, it suffices to do the following:
   (a) replace every sheet of the scheme of the Riemann surface of the function $f(z)$ by a pack of n sheets;
   (b) turning around an arbitrary branch point of the function $h(z)$ one moves from all sheets of one pack to all sheets of a different pack;
   (c) these passages from one pack of sheets to another correspond to the passages between the sheets of the Riemann surface of the function $f(z)$;
   (d) if the branches in the bunches are enumerated in such a way that $f_{i,k}(z) = f_{i,0}(z)\epsilon_n^k$, then by moving from one bunch to another the sheets of the corresponding packs are not mixed but they permute cyclically!

(23) Suppose that a path C on the z plane avoids the branch points and the non-uniqueness points of the functions $w(z)$. Prove that moving along the path C, starting from distinct sheets of the scheme of the Riemann surface of function $w(z)$, arrives at distinct sheets.

So, to each turn (counter-clockwise) around any branch point of the function $w(z)$ there corresponds a permutation of the sheets of the schemes of the Riemann surface of the function $w(z)$. This leads us to the following definition.

Definition 5. Let $g_1, \ldots, g_s$ be the permutations of the sheets of the scheme of a Riemann surface corresponding to the turns (counterclockwise) around all the branch points. We call the subgroup generated by the elements $g_1, \ldots, g_s$ the **permutation group of the sheets of the given scheme of the Riemann surface**, shortened as the **permutation group of the given scheme**.

(24) (1 pt) What are the permutation groups of the schemes of the following functions: a) $\sqrt{z}$; b) $\sqrt[n]{z}$ c) $\sqrt[3]{z^2-1}$ d) $\sqrt[4]{(z-1)^2(z+1)^3}$?

(25) (6 pts) Compute the permutation groups of the schemes of the following functions: a) $\sqrt{z} + \sqrt{z-1}$ b) $\sqrt[3]{z^2-1} + \sqrt{1/z}$ c) $\sqrt{z-1}.\sqrt[4]{z}$ d) $\sqrt{z^2-1}/\sqrt[4]{z+1}$ e) $(\sqrt{z} + \sqrt{z})/\sqrt[3]{z(z-1)}$ f) $(\sqrt{z}.\sqrt[3]{z-1})^3$.

Suppose that the point z_0 is neither a branch point nor a non-uniqueness point of the multi-valued function $w(z)$, and that $w_1, w_2, \ldots, w_n$ are all values of the function $w(z)$ at the point z_0. Consider a continuous path C starting and ending at the points z_0 (i.e. a loop based at z_0) and not passing through any branch point and any non-uniqueness point of the function $w(z)$. Take a value $w_i = w(z_0)$ and define by continuity along the path C a new value $w_j = w(z_0)$. Starting from distinct values w_i we obtain different values w_j. Hence to the path C, there corresponds a certain permutation of $w_1, \ldots, w_n$, say g. Also note that to the reverse path C^{-1}, we can associate the permutation g^{-1} and to the concatenation of two paths, we can associate the permutation g_2g_1. Thus, if we consider all possible loops based at z_0, then the corresponding permutations will form a group, the group of permutations of the values $w(z_0)$.

(26) (2 pts) Let G_1 be the permutation group of the values $w(z_0)$ and G_2 the permutation group of some scheme of the function $w(z)$. Prove that the groups G_1 and G_2 are isomorphic.

Note that in the definition of the permutation group of the values $w(z_0)$ one does not use any scheme of the Riemann surface of the function. Thus from the above exercise it follows that the permutation group of the values $w(z_0)$ for an arbitrary point z_0 and the permutation group of any arbitrary scheme of the Riemann surface of the function $w(z)$ are isomorphic. Consequently, the permutation group of the values $w(z_0)$ for all points z_0 and the permutation group of all the schemes of the Riemann surfaces of the function $w(z)$ are isomorphic, i.e., they represent one unique group. This group is called the **monodromy group of the multi-valued function w(z)**.

(27) (1 pt) Show that the above definition is a special case of the monodromy representation defined in the course.

Let us prove the following theorem using the next seven exercises:

Theorem 6. *If the multivalued function $h(z)$ is representable by radicals, then its monodromy group is soluble.*

(28) (4 pts) Let $h(z) = f(z) + g(z)$ (resp. $f(z) - g(z)$, $f(z).g(z)$, $f(z)/g(z)$) and suppose that we have built the scheme of the Riemann surface of the function $h(z)$ by the method as proved above. Prove that if F and G are the permutation groups of the initial schemes, then the permutation groups of the scheme built admits a surjective map from a subgroup of the direct product $G \times F$.

Hint: First show that the permutation group for the scheme built using only part a) of Ex. 20 (i.e, if we don't identify the identical sheets) is a subgroup of the direct product and then show that there is a surjective map from this group to the permutation group of the actual scheme of $h(z)$.

(29) (1 pt) Suppose the monodromy groups of the functions $f(z)$ and $g(z)$ be soluble. Prove that the monodromy groups of the functions $h(z) = f(z)+g(z)$, $h(z) = f(z).g(z)$, $h(z) = f(z)/g(z)$ are soluble as well.

(30) (2 pts) Suppose the monodromy group of the function $f(z)$ is soluble. Prove that the monodromy group of the function $h(z) = [f(z)]^n$ is also soluble.

(31) (2 pts) Let H be the permutation group of a scheme of the function $h(z) = \sqrt[n]{f(z)}$ and F the permutation group of a scheme of the function $f(z)$, made with the same cuts. Define a surjective homomorphism of the group H onto the group F.

(32) (1 pt) Prove that the kernel of the homomorphism defined in the exercise above is abelian.

(33) (1 pt) Suppose the monodromy group F of the function $f(z)$ be soluble. Prove that the monodromy group H of the function is also soluble.

Thus, we have proved the theorem above.

Now consider the equation

$$3w^5 - 25w^3 + 60w - z = 0. \tag{1}$$

where z is treated as a parameter and for every complex value of z we look for all complex roots w of this equation. From the fundamental theorem of algebra, the given equation for every z has 5 roots, not necessarily distinct. One can easily check that the above equation has 4 distinct roots for $z = \pm 38$ and $z = \pm 16$, and for the other values of z it has 5 distinct roots.

(34) (4 pts) Let z_0 be an arbitrary complex number and w_0 be one of the roots of equation above for $z = z_0$. Consider a disc of radius r arbitrarily small with its center at w_0. Prove that there exists a real number $\rho > 0$ such that if $|z_0' - z_0| < \rho$ then in the disc considered there exists at least one root of the equation above for $z = z_0'$ also.

Suppose the function $w(z)$ expressed the roots of the equation 1 in terms of the parameter z.

(35) (4 pts) Prove that points different from $z = \pm 38$ and $z = \pm 16$ can neither be branch points nor non-uniqueness points for $w(z)$. Now $w(z)$ being an algebraic function possess the monodromy property. One can therefore build for the function $w(z)$ the Riemann surface. This Riemann surface evidently has 5 sheets.

(36) (2 pts) Suppose it is known that the point $z_0 = 38$ (resp. $z_0 = -38$ or ± 16) is a branch point of the function $w(z)$ expressing the roots of equation 1 in terms of the terms of the parameter z. How do the sheets of the Riemann surface of the function $w(z)$ at the point z_0 (more precisely, along the cuts joining the point z_0 to infinity) join?

(37) (2 pts) Let z_0 and z_1 be two arbitrary points different from $z = \pm 38$ and ± 16, and w_0 and w_1 be two arbitrary images of these points under the mapping $w(z)$. Prove that it is possible to draw a continuous curve joining the points z_0 and z_1, not passing through the points $z = \pm 38$ and ± 16 and such that its continuous image, starting from the point w_0, ends at the point w_1.

(38) (3 pts) Prove that all four points $z = \pm 38$ and ± 16 are branch points of the function $w(z)$. How can we represent the scheme of the Riemann surface of the function w(z)? Draw all different possibilities schemes (we consider different two schemes if they cannot be obtained one from another by a permutation of the sheets and of the branch points).

(39) (2 pts) Find the monodromy group of the function $w(z)$ expressing the roots of the equation

$$3w^5 - 25w^3 + 60w - z = 0$$

in terms of parameter z.

(40) (1 pt) Prove that the function $w(z)$ is not representable by radicals.

(41) (1 pt) Prove that the algebraic general equation of 5th degree

$$a_0w^5 + a_1w^4 + a_2w^3 + a_3w^2 + a_4w + a_5 = 0$$

where a_i are complex parameters and $a_0 \neq 0$, is not solvable by radicals i.e., that there are no formulae expressing the roots of this equation in terms of the coefficients by means of the operations of addition, subtraction, division, elevation to an integer power and extraction of a root of integer order.

(42) (2 pts) Consider the equation

$$(3w^5 - 25w^3 + 60w - z)w^{n-5} = 0$$

and prove that for $n > 5$ the general algebraic equation of degree n is not solvable by radicals.

Thus you have proved:

Theorem 7 (Abel's Theorem)**.** *For $n \geq 5$, the general algebraic equation of degree n*

$$a_0w^n + a_1w^{n-1} + \ldots a_{n-1}w + a_n = 0$$

is not solvable by radicals.