EXERCISE SHEET 8: ALGEBRAIC CURVES AND RIEMANN SURFACES

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1. World of Riemann Surfaces-from Miranda's book

(1) (5 pts) Let X be the Riemann Sphere $\mathbb{C}_\infty$, with local coordinate z in one chart and $w = 1/z$ in the other chart. Let ω be a meromorphic 1-form on X. Show that if $\omega = f(z)dz$ in the coordinate z, then f ust be a rational function of z. Show further that there are no non zero holomorphic 1-forms on $\mathbb{C}_\infty$. Where are the zeroes and poles, and the other orders of the meromorphic 1-form defined by dz? Of the form dz/z?

(2) (4 pts) Let L be a lattice in $\mathbb{C}$, and let $\pi : \mathbb{C} \to X = \mathbb{C}/L$ be the natural quotient map. show that the local formula dz in every chart of $\mathbb{C}/L$ is a well defined holomorphic 1-form on $\mathbb{C}/L$. Show that the local formula $d\bar{z}$ in every chart of $\mathbb{C}/L$ is a well defined C^∞ 1-form on $\mathbb{C}/L$.

(3) (4 pts) Let X be a smooth affine plane curve defined by $f(u, v) = 0$. Show that du and dv define holomorphic 1-forms on X, as do $p(u, v)du$ and $p(u, v)dv$ for any polynomial $p(u, v)$. Show that if $r(u, v)$ is any rational function, then $r(u, v)du$ and $r(u, v)dv$ are meromorphic 1-forms on X. Show that $(\partial f/partialu)du = -(\partial f/partialv)dv$ as holomorphic 1-forms on X.

(4) (2 pts) Suppose that X is a projective plane curve of degree d with nodes, defined by the affine equation $f(u, v) = 0$. Show that if $p(u, v)$ is any polynomial of degree at most $d-3$, which vanishes at the nodes of X, then $p(u, v)du/(\partial f/\partial v)$ defines a holomorphic 1-form on the resolution $\tilde{X}$ of the nodes.

(5) (2 pts) Let L be a lattice in $\mathbb{C}$, and let $\pi : \mathbb{C} \to X = \mathbb{C}/L$ be the natural quotient map. Show that $dz \wedge d\bar{z}$ is a well defined C^∞ 2-form on $\mathbb{C}/L$.

(6) (3 pts) Let a holomorphic map $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by the formula $w = z^N$ for some integer $N \geq 2$, where we use z as an affine coordinate in the domain and w as an affine coordinate in the range. Compute the pullback $F * (dw)$ of the form $(1/w)dw$. Compute the orders of $F^*(dw)$ at all of its zeroes and poles.

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