EXERCISE SHEET 4: ALGEBRAIC CURVES AND RIEMANN SURFACES

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1. World of Riemann Surfaces-from Miranda's book

(1) (4 pts) Let G act continuously on a topological space X, and let Y be a topological space. Show that a map $\alpha : X/G \to Y$ is continuous if and only if $\alpha \circ \pi : X \to Y$ is continuous and G-invariant. Show that there is a 1-1 correspondence between

$$\left\{\begin{matrix}\text{Continuous maps} \\ \alpha : X/G \to Y\end{matrix}\right\} \text{ and } \left\{\begin{matrix} G-\text{invariant} \\ \text{continuous maps} \\ \beta : X \to Y\end{matrix}\right\}$$

which associates to α the map $\beta = \alpha \circ \pi$.

(2) (4 pts) Show that the group of automorphisms of $\mathbb{C}_\infty$ generated by the two automorphisms sending z to $exp(2\pi i/r)z$ and sending z to $1/z$ is a dihedral group of order 2r, which acts holomorphically and effectively on $\mathbb{C}_\infty$. Show that there are three branch points to the quotient map, with ramification numbers $2, 2, r$.

(3) (4 pts) Show that the "Klein curve" X defined by $xy^3 + yz^3 + zx^3 = 0$ is a smooth projective plane curve. Since it has degree 4, X has genus 3. Show that it realizes the Hurwitz bound by finding 168 automorphisms.

(4) (2 pts) Suppose that a holomorphic map $F : X \to \mathbb{P}^1$ of degree d is defined by choosing a set of branch points $\{b_1, \dots, b_n\}$ in $\mathbb{P}^1$ and a corresponding set of permutations $\sigma_1, \dots, \sigma_n$ in S_d, which generate a transitive subgroup of S_d and whose product is 1. Suppose that the permutation σ_i is a product of k_i disjoint cycles. Show that the genus g of the compact Riemann surface X is

$$g = 1 + \frac{(n-2)d - \Sigma_{i=1}^n k_i}{2}.$$

(5) (4 pts) Let $f(z) = z^3/(1-z^2)$ define a holomorphic map of degree 3 from $\mathbb{P}^1$ to itself. Find all of the branch points, and the corresponding permutations in S_3.

(6) (4 pts) Let X denote the Fermat curve of degree d in $\mathbb{P}^2$, defined by homogeneous polynomial $x^d + y^d + z^d = 0$. Let $F : X \to \mathbb{P}^1$ be defined by $F([x : y : z]) = [x : y]$. Show that F has d branch points, and find the d corresponding permutations.

(7) (3 pts) Let G be the dihedral group of order $2r$ acting on $\mathbb{P}^1$, with three branch points $\{b_1, b_2, b_3\}$ for the quotient map $\pi : \mathbb{P}^1 \to \mathbb{P}^1$; moreover assume that for each $i = 1, 2, 3$, the map π has multiplicity r_i at each of $2r/r_i$ points lying above b_i, with $\{r_i\} = \{2, 2, r\}$. Find the three corresponding permutations in S_{2r}.

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(8) (5 pts) Show that the fundamental group of a compact Riemann surface X of genus g has a presentation of the form :

$$\pi_1(X, x) = \langle a_1, b_1, \dots, a_g, b_g | [a_1, b_1] \dots [a_g, b_g] = 1 \rangle,$$

where the brackets $[a_i, b_i]$ denote the commutators $a_i b_i a_i^{-1} b_i^{-1}$. Therefore an unramified covering $F : X \to Y$ of degree two is determined by giving $2g$ permutations in S_2 satisfying the above relation, which generate a transitive subgroup. For the permutations to generate a transitive subgroup is easy: not all the permutations should be identity. Show that the number of non-isomorphic unramified coverings of Y is $2^{2g} - 1$. In case, $g = 1$, assume that Y is a complex torus given by a lattice L in $\mathbb{C}$; find the three sublattices of L corresponding to the three non-isomorphic covers.