Theorem (Classification of Surfaces): Any compact surface is homeomorphic to a sphere or to a connected sum of tori or to a connected sum of projective planes.

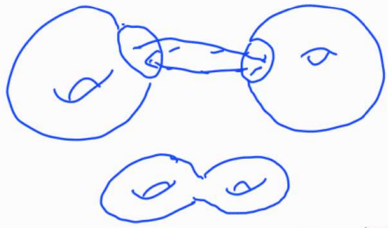
Orientability

Euler-Poincaré characteristic (Tomorrow)

Recall connected sum:

Let S_1 & S_2 be disjoint surfaces.

connected sum: $S_1 \# S_2$



Precisely, we choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ s.t D_1 & D_2 are closed discs

Let $S_i' = S_i \setminus D_i \quad i = 1, 2$

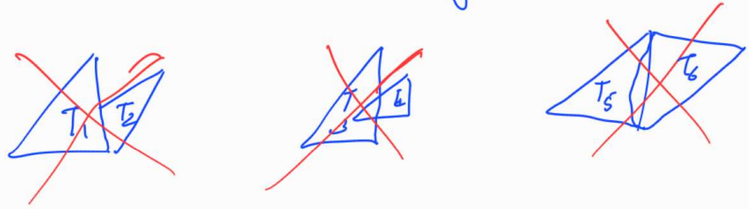
Choose a homeomorphism h of the boundary circle of D_1 onto the boundary of D_2

Then $S_1 \# S_2 = \dfrac{S_1' \cup S_2'}{\langle x \sim h(x),\ \forall x \in D_1 \rangle}$

- Clearly, $S_1 \# S_2$ is a surface.
- Check: The homeomorphism type of $S_1 \# S_2$ does not depend upon the choice of D_1 & D_2, nor on the choice of h.

Def^n: A triangulation of a compact surface S consists of a finite family of closed subsets $\{T_1, \ldots, T_n\}$ that cover S, and a family of homeomorphisms $\varphi_i : T_i' \to T_i$, $i = 1, \ldots, n$ where each T_i' is a triangle in the plane $\mathbb{R}^2$ (i.e. a compact subset of $\mathbb{R}^2$ bounded by three distinct straight lines).

- The subsets T_i are called triangles
- The subsets of T_i that are the images of vertices & edges of the triangles T_i' under φ_i are called "vertices" & "edges". resp.
- It is required that any two distinct triangles T_i & T_j either be disjoint or have a single vertex in common or have an entire edge in common.

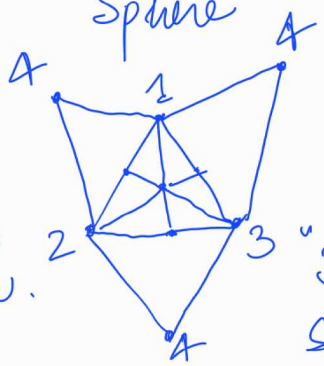


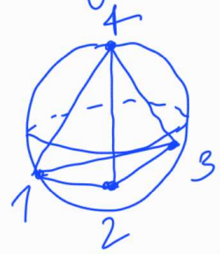
Triangulated surface is a surface constructed by gluing Δ's.

- Because two different Δ's cannot have the same vertices we can specify completely a triangulation by numbering the vertices and then listing which triples of vertices are vertices of a triangle.
- Such a list of Δ's completely determines the surface together with the given Δgulation upto homeomorphism.

Eg:

Sphere





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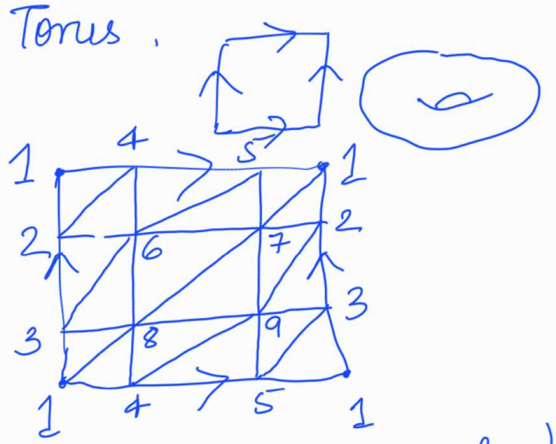
triangulation of a sphere.

"Barycentric subdivision"

Torus.



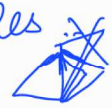
Check: Nothing less works!

Thm: (Rado) Every compact (connected) surface has a triangulation!

[We will prove this for Riemann surfaces].

Remarks: Any triangulation of a compact surface satisfies the following 2 conditions:

(1) Each edge is an edge of exactly two triangles.



[Each pt. on the edge has a nbhd homeo to $\mathbb{R}^2$.

& 3-triangles have the same edge $\to \mathbb{R}^3$

1-triangle $\to$ edge $\to$ end of the world $\to$ 1 dim'l $\to$ upper half plane.

(2) Let v be a vertex of a triangulation. Then we may arrange the set of all triangles with v as a vertex in cyclic order $T_0, T_1, T_2, \ldots, T_{n-1}, T_n = T_0$, such that T_i and T_{i+1} have an edge in common for $0 \le i \le n-1$.

- Easy consequence of (1) is that the set of all triangles with v as a vertex can be divided into several disjoint subsets, such that the triangles in each subset can be arranged in cyclic order.
- However, if there were more than one such subset then the nbhd of v would not be homeo. to $\mathbb{R}^2$.

<u>Back to R.S</u>:

We want to construct Δ of R.S, we will use meromorphic functions.

<u>Ex. sheet 5</u>: Let f be a meromorphic function on X. We define $F: X \longrightarrow \mathbb{C}_\infty$ by

$$F(x) = \begin{cases} f(x) \in \mathbb{C} & \text{if } x \text{ is not a pole of } f \\ \infty & \text{if } x \text{ is a pole of } f. \end{cases}$$

Then F is a holomorphic map!

$$\left\{ \begin{array}{l} \text{meromorphic } f^n \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{holomorphic maps} \\ F: X \to \mathbb{C}_\infty \cong \mathbb{P}^1 \\ \text{which are not identically } \infty \end{array} \right\}$$

$$\text{constant } f^n \longmapsto \text{constant maps (not to } \infty)$$

<u>Local Normal form of a Holomorphic map</u>:

<u>Slogan</u>: Every holomorphic map looks like a power map locally.

<u>Prop^n</u>: (Local normal form) Let $F: X \to Y$ be a holomorphic map defined at $p \in X$, which is not constant. Then there is a unique integer $m \ge 1$ which satisfies the following property: for every chart $\phi_2: U_2 \to V_2$ on Y centered at $F(p)$,

for every chart $\phi_2 : U_2 \to V_2$

$\exists$ a chart $\phi_1 : U_1 \to V_1$ on X centered at p. s.t.

$$\phi_2(F(\phi_1^{-1}(z))) = z^m.$$

$$\begin{tikzcd}
F: X \arrow[r] & Y \\
\exists\, U_1 \ni p \arrow[d, "\phi_1"'] & U \ni F(p) \\
& U_2 \ni \arrow[d, "\phi_2"] \\
V_1 \arrow[uuu, bend left, "\phi_1^{-1}"] & V_2
\end{tikzcd}$$

$$z \longmapsto z^m$$

Proof: Fix a chart ϕ_2 on Y centered at $F(p)$ and choose any chart $\psi : U \to V$ on X centered at p.

Then the Taylor series for the function $T(w) = \phi_2(F(\psi^{-1}(w)))$ must be of the form $T(w) = \sum_{i=m}^{\infty} c_i w^i$.

with $c_m \neq 0$ and $m \geq 1$ since $T(0) = 0$.

Then we have $T(w) = w^m S(w)$

where $S(w)$ is a holomorphic f^n at $w = 0$ & $S(0) \neq 0$.

In this case there exist a function $R(w)$ near 0 s.t.

$$R(w)^m = S(w),$$

$$\Rightarrow T(w) = (wR(w))^m.$$

Let $\eta(w) = wR(w)$; since $\eta'(0) \neq 0$. we see that near 0 the function η is invertible (by IFT) and of course holomorphic

Hence the composition $\phi_1 = \eta \circ \psi$ is a also a chart on X defined and centered near p.

Think of η as defining a new coordinate z (via $z = \eta(w)$) we see that z and w are related by $z = wR(w)$.

Thus,
$$\phi_2(F(\phi_1^{-1}(z))) = \phi_2(F(\psi^{-1}(\eta^{-1}(z))))$$
$$= T(\eta^{-1}(z))$$
$$= T(w) = (wR(w))^m$$
$$= z^m.$$

The uniqueness of m comes from noticing that if there are local coordinates at p & $F(p)$ s.t the map F has the for $z \mapsto z^m$ then near p, there are exactly m preimage of points near $F(p)$.

Thus, this exponent m can be detected solely by studying the topological properties of the map F near p and is therefore independent of the choice of local coordinates. $\square$.

<u>Defn</u>: The <u>multiplicity</u> of F at p, denoted $\text{mult}_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \mapsto z^m$.

<u>Example</u> (trivial): $\phi : U \to V$ chart map for X, considered as a holomorphic map to $\mathbb{C}$.
Then $\text{mult}_p(\phi) = 1$ $\quad p \in U$. [Bijective]

Remark
- $\text{mult}_p(F) \geq 1$.

(Simpler way to compute)
Take any local coordinates z near p and w near $F(p)$
say $p \rightsquigarrow z_0$ and $F(p) \rightsquigarrow w_0$

In terms of z, w, the map F may be written as $w = h(z)$ where h is holomorphic. Then $w_0 = h(z_0)$

<u>lemma 13.1</u> (4.4): With the above notation, the multiplicity of F at p is
$$\text{mult}_p(F) = 1 + \text{ord}_{z_0}(dh/dz).$$

In particular, the multiplicity is the exponent of lowest strictly +ve term of the power series for h: if $h(z) = h(z_0) + \sum_{i=m}^{\infty} c_i (z-z_0)^i$, $m \geq 1$ and $c_m \neq 0$, then $\text{mult}_p(F) = m$.

<u>Pf</u>: Previous pf the $\text{mult}_p(F)$ was the lowest term appearing in the power series T for F when centered local coordinates are used at p and the image $F(p)$:

$z - z_0$ and $w - w_0$ are centered local coordinates

since $w - w_0 = h(z) - h(z_0)$
about $z = z_0$. mult_p = lowest term

Taylor's theorem $\Rightarrow$ <u>this</u> is one more that the order of the derivative of h at z_0. $\square$

- The above lemma shows that the pts. of the domain where F has multiplicity at least 2 form a discrete set!

Pf: pt. with $\mathrm{mult}_p F \geq 2$.
these are pts. are zeroes of derivative of a local formula h for F, and since h is holomorphic, the zeroes of its derivative are discrete.
X compact $\Rightarrow \{p \mid \mathrm{mult}_p F \geq 2\}$ is finite.

Defn: Let $F: X \to Y$ be a non-constant holomorphic map.
A point $p \in X$ is a ramification pt. for F if $\mathrm{mult}_p(F) \geq 2$.
A point $y \in Y$ is a branch point for F if it is the image of a ramification pt. for F.

The ramification and branch points for a holomorphic map form discrete subsets of the domain & range resp.

Example: Suppose X is a smooth affine plane curve defined by $f(x,y) = 0$.
Define $\pi: X \to \mathbb{C}$ by projection onto the x-axis
$(x,y) \longmapsto x$ (holomorphic)

Claim: π is ramified at $p = (x_0, y_0) \in X$ iff $\left(\frac{\partial f}{\partial y}\right)(p) \neq 0$.

Pf: Assume first $(\partial f/\partial y)(p) \neq 0$
$\Rightarrow$ π is a chart map for X near p. and so $\mathrm{mult}_p \pi = 1$.

Conversely, suppose $(\partial f/\partial y)(p) = 0$.
Then since X is smooth at p $(\partial f/\partial x)(p) \neq 0$.

& so y is a chart map for X near p.
By IFT $\Rightarrow$ X is locally the graph of a holomorphic fn $g(y)$.
∂f

Hence $f(g(y), y) \equiv 0$ in a nbd of y_0. ∂y

Take the derivative wrt y

$$\Rightarrow \frac{\partial f}{\partial x} g'(y) \pm \frac{\partial f}{\partial y} \equiv 0 \text{ in a nbd of } p.$$

$\frac{\partial f}{\partial x} \neq 0$; $\frac{\partial f}{\partial y} = 0$, at p.

$$\left(\frac{\partial f}{\partial x}\right)(p)\, g'(y_0) = 0$$

$\frac{\partial f}{\partial x}(p) \neq 0$ Smoothness. $g'(y_0) = 0$.

$\Rightarrow g'(y_0) = 0$.

But $g(y)$ is exactly the local formula for the map π.

Hence by the derivative criterio of lemma 13.2 (**)

π is ramified at p.

- ~~Same~~ Similar holds for smooth projective plane curves X.

$\square$.