EXERCISE SHEET 6: ALGEBRAIC CURVES AND RIEMANN SURFACES

TANYA KAUSHAL SRIVASTAVA

DUE DATE: 23 November 2021

1. Triangulation

(1) (2 pts) Give a triangulation of compact Mobius strip and compute its Euler number.
(2) (2 pts) Give a triangulation for the real projective plane and compute its Euler number.

2. Exercises on Riemann Surfaces- from the Book- Rick Miranda

(1) (4 pts) Let $f(z) = z^3/(1-z^2)$, considered as a meromorphic function on the Riemann sphere $\mathbb{C}_\infty$. Find all the points p such that $ord_p(f) \neq 0$. Consider the associated map $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Show that F has degree 3 as a holomorphic map, and find all its ramification and branch points. Verify Hurwitz's formula for this map F.
(2) (4 pts) Let $f(z) = 4z^2(z-1)^2/(2z-1)^2$, considered as a meromorphic function on the Riemann sphere $\mathbb{C}_\infty$. Find all points p such that $ord_p(f) \neq 0$. Consider the associated map $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Show that F has degree 4 as a holomorphic map, and find all its ramification and branch points. Verify Hurwitz's formula for this map F.
(3) (4 pts) Let $F : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces:
   (a) Show that if $Y \cong \mathbb{P}^1$ and F has degree at least 2, then F must be ramified.
   (b) Show that if X and Y both have genus one, then F is unramified.
   (c) Show that $g(Y) \leq g(X)$ always.
   (d) Show that if $g(Y) = g(X) \geq 2$, then F is an isomorphism.
(4) (4pts) Let X be the projective plane curve of degree d defined by the homogeneous polynomial $F(x, y, z) = x^d + y^d + z^d$. This is the **Fermat curve** of degree d. Let $\pi : X \to \mathbb{P}^1$ be given by $\pi[x : y : z] = [x : y]$.
   (a) Check that the Fermat curve is smooth.
   (b) Show that π is a well defined holomorphic map of degree d.
   (c) Find all ramification and branch points of π.
   (d) Use Hurwitz's formula to compute the genus of the Fermat curve:
$$g(X) = \frac{(d-1)(d-2)}{2}.$$
(5) (3 pts) Let U be the plane affine curve defined by $x^2 = 3+10t^4+3t^8$. Let V be the affine plane curve defined by $w^2 = z^6 - 1$. Show that both curves are smooth. Show that the function $F : U \to V$ defined by $z = (1+t^2)/(1-t^2)$ and $w = 2tx/(1-t^2)^3$ is holomorphic and nowhere ramified whenever $t \neq \pm 1$.

Date: November 12, 2021.

(6) (2 pts) Now let X (resp. Y) be the compact hyperelliptic curve defined by $x^2 = 3+10t^4+3t^8$. (resp. $w^2 = z^6-1$). Let U and V be the corresponding affine plane curves, which are complements in X and Y respectively of the points at infinity. Show that the function $F : U \to V$ defined by $z = (1+t^2)/(1-t^2)$ and $w = 2tx/(1-t^2)^3$ extends to a holomorphic map from X to Y of degree 2, which is nowhere ramified. What is the genus of X and Y?