Holomorphic maps between Riemann Surfaces

Recall: We defined the category of Riemann surfaces.

- Composition of holomorphic map with a holomorphic function is a holomorphic function
- Composition of holomorphic map with a meromorphic function is a meromorphic function (* image of $F(X)$ must not be a subset of the set of poles of g)

Let rephrase these properties:

Let $F: X \to Y$ be a holomorphic map b/w R.S. Then for every open set $W \subset Y$. F induces a $\mathbb{C}$-algebra homomorphism

$$F^*: \mathcal{O}_Y(W) \longrightarrow \mathcal{O}_X(F^{-1}(W))$$

ring of holomorphic functions on $F^{-1}(W)$

defined by the composition

$$g \longmapsto F^*(g) = g \circ F$$

For meromorphic functions

$$F^*: \mathcal{M}_Y(W) \longrightarrow \mathcal{M}_X(F^{-1}(W)).$$

$$h \longmapsto h \circ F$$

if F is not constant.

If $F: X \to Y$ & $G: Y \to Z$ are holomorphic maps,

$$F^* \circ G^* = (G \circ F)^*.$$

When are two Riemann surfaces same?

Defn: An isomorphism (or biholomorphism) between R.S is a holomorphic map $F: X \to Y$ which is bijective & whose inverse $F^{-1}: Y \to X$ is holomorphic

A self-iso $F: X \to X$ is called an automorphism.

If $\exists$ an iso b/w X & Y, we say X & Y are isomorphic (biholomorphic)

Ex: The Riemann sphere $\mathbb{C}_\infty$ and $\mathbb{CP}^1$, complex projective line

Ex: The ... of
are biholomorphic.

- If X & Y are isomorphic
$\Rightarrow$ $\mathcal{O}_Y(W) \simeq \mathcal{O}_X(F^{-1}(W))$ for any iso $F: Y \to X$
$\mathcal{M}_Y(W) \simeq \mathcal{M}_X(F^{-1}(W))$ $W \subset Y$ open

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Extremely Easy theorems

Propⁿ: (Open Mapping Theorem) Let $F: X \to Y$ be a non-constant holomorphic map between R.S. Then F is an open mapping.

Propⁿ: Let $F: X \to Y$ injective (1-1) holomorphic map b/w R.S. Then F is an iso b/w X & $F(X)$.

Propⁿ: (Identity thm) Let F & G be two holomorphic maps b/w X & Y. If $F = G$ on a subset S of X with a limit pt. in X, then $F = G$.

Topological theorems

Propⁿ: Let X be a compact R.S. and $F: X \to Y$ be a non-constant holomorphic map. Then Y is compact & F is onto.

Pf: F is holomorphic $\overset{\text{OMT}}{\Rightarrow}$ F is open
$\Rightarrow$ $F(X) \subset Y$ open

X compact $\underset{+F\text{ cont}^n}{\Rightarrow}$ $F(X)$ compact.

Y Hausdorff $\Rightarrow$ $F(X)$ is closed

Y connected $\Rightarrow$ $F(X) = Y$. ∎

Propⁿ: (Discreteness of Preimage): Let $F: X \to Y$ be a non-constant holomorphic map b/w R.S. Then for every $y \in Y$, the preimage $F^{-1}(y)$ is a discrete subset of X. In particular, if X & Y are compact, then $F^{-1}(y)$ is a non-empty finite set for every $y \in Y$.

Pf: Fix a local coordinate z centered at $y \in Y$ and for a point $x \in F^{-1}(y)$ choose another local coord. w centered at

Then F written in terms of the local coordinates.
is a non-constant holo. f^n $z = g(w)$
g has zero at the origin.
since x (which is $w=0$) goes to y (which is $z=0$).
$g(0)=0$.

Zeros of non-constant holomorphic f^n are discrete.
we see that is some nbhd of x, x is the only preimage of y. $\Rightarrow$ $F^{-1}(y)$ is a discrete subset of X.

Second statement, X compacts $\Rightarrow$ F is onto
& discrete subsets of compact space are finite. $\square$

Classification of surfaces : (topological 2-manifold).

Thm: Two compact surfaces are homeomorphic iff they agree in character of orientation / orientability and Euler-Poincare characteristic ($\Leftrightarrow$ genus).
$\Updownarrow$
holes in your surface.

g holes orientable

0 holes

Every proof of classification theorem has 2 main steps

1. A topological step: Shows that every compact surface can be triangulated.
(combinatorial info)

2. A combinatorial step: show that every triangulated surface can be converted into a normal form of a surface. (distinguished by a numerical invariant).

Recall: Orientable vs. non-orientable manifolds.
... coordinate system

- On $\mathbb{R}^2$ a. left handed coordinate system
right handed —

b. clockwise direction of rotation 
anti clockwise

- $\mathbb{R}^2$ orientable

- Go on a surface - its locally like $\mathbb{R}^2$
(Ant cn) pick an orientation for each pt. on the surface.

Mobius band - non-orientable surface

$X = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 < y < 1\}$

$Y = X / ((-1,-y) \sim (1,y))$.

Ex: Another example of a non-orientable surface is $\mathbb{RP}^2$.

Let us define orientation formally:

Let $\varphi : D \to \mathbb{C}$ be a cont^n function.
$D \subseteq \mathbb{C}$

We say that φ is regular at $z_0 \in D$ if there is some set $V \subseteq D$ containing z_0 such that $\varphi(z) \neq \varphi(z_0)$ $\forall z \in V$.

Eg [injective functions]

Assuming that φ is regular at z_0, we will define the degree of φ at z_0.

Let Ω be a punctured open disk $\{z \in V \mid |z - z_0| < r\}$ contained in V.

Since φ is regular at z_0, it maps Ω into the punctured

Since φ is reg ... $w_0 = \varphi(z_0)$ in $\mathbb{C}$.
plane Ω' obtained by deleting $w_0 = \varphi(z_0)$ in $\mathbb{C}$.

Now, φ induces a homomorphism

$$\varphi_*: \pi(\Omega) \longrightarrow \pi(\Omega'), \text{ fundamental gps of } \Omega \,\&\, \Omega'.$$

$$\begin{array}{ccc} \pi(\Omega) & & \pi(\Omega') \\ \wr\!\shortparallel & & \wr\!\shortparallel \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

$[\alpha]$ $\quad d$ +ve
$\quad\quad$ -ve

α is some circle centered at a

β some circle centered at $\varphi(a)$.

$$[\alpha] \longmapsto [\beta^d] \qquad d \in \mathbb{Z}$$

If $d = 0$ then $\varphi_*(\pi(\Omega)) = 1$

if $d \neq 0$ $\quad \varphi_*(\pi(\Omega)) =$ infinite cyclic group generated by the class of β^d.

Define: d to be the degree of φ at z_0. and we denote it as $d(\varphi)_{z_0}$.

Ex: Prove that this definition does not depend on the choice of a (the center of the circle α) in Ω and thus does not depend on Ω.

- If we have another map ψ regular at $w_0 = \varphi(z_0)$, then $\psi \circ \varphi$ is regular at z_0 and it is immediately verified that
$$d(\psi \circ \varphi)_{z_0} = d(\psi)_{w_0}\, d(\varphi)_{z_0}.$$

D (connected open set) and φ is a homeomorphism between D and $\varphi(D)$.

then we have $\varphi(D)$ is also open and thus we can define the degree of the inverse mapping φ^{-1}. and since identity clearly has deg 1 we get $d(\varphi)\, d(\varphi^{-1}) = 1$

$$\Rightarrow d(\varphi)_{z_0} = \pm 1$$

Prop^n: Given a region, D, in a plane, for every homeo φ b/w D and $\varphi(D)$, if $\varphi(D)$ has a non-empty interior, then the degree $d(\varphi)_z$ is constant for all $z \in D$ and in fact, $d(\varphi) = \pm 1$.

Def^n: When $d(\varphi) = 1$ in Prop^n above, we say that φ is sense-preserving and when $d(\varphi) = -1$ we say that φ is sense-reversing.

Orientability of a surface:

Given a surface we will call a region V on it a planar region if there is a homeomorphism $h: V \to U$ open $\subseteq \mathbb{C}$ or $\mathbb{R}^2$.

From the prop^n above, the homeomorphism $h: V \to U$ can be divided into 2 classes:

2 homeomorphism h_1, h_2 are equivalent iff $h_1 \circ h_2^{-1}$ has degree 1 i.e. sense preserving.

Observation: $h: V \to U \subseteq \mathbb{C}$.

$\bar{h}: V \to \mathbb{C}$

$z \mapsto \bar{h}(z) = \overline{h(z)}$ complex conjugation.

Then h & $\bar{h}$ are not equivalent.

$d(h \circ \bar{h}^{-1}) = -1$.

For any map $g: V \to U \subset \mathbb{C}$

$h \circ g^{-1}$ or $\bar{h} \circ g^{-1}$ is sense preserving.

Hence there are exactly 2 equivalence classes of homeomorphism.

... choice ... one of the two classes of homeomorphism

Def^n: The choice of one of ... h as above constitutes an orientation of V.

- An orientation of V induces an orientation on any subregion W of V by restriction.

If V_1 & V_2 are two planar regions and these regions have their resp. orientations, we say that these orientations are compatible if they induce the same orientation on all common subregions of V_1 & V_2.

Def^n: A surface F is orientable if it is possible ~~to~~ to assign an orientation to all planar regions in such a way that the orientation of any 2 overlapping regions are compatible.

Thm/Ex: All Riemann surfaces are orientable! $\square$

$h_1 \circ h_2^{-1}$ are holomorphic maps

$\Rightarrow$ sense preserving.