Announcement: Quiz Take Home
Next Week - On moodle
(Approx. 36 hours to finish it)

Meromorphic functions on Riemann Surfaces

$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^*$ ⇝ ? Rational functions.

$\lambda \in \mathbb{C}^* \curvearrowright \mathbb{C}^2 \setminus \{0\}$
$(z, w) \mapsto (\lambda z, \lambda w)$

[When are 2 R.S. "same"?]
"How to check they are same?"

? $\updownarrow$

$\mathbb{C}_\infty$ Riemann Sphere

Strategy: 1) Construct functions on $\mathbb{C}^2$ which are invariant under the action of $\mathbb{C}^*$, so they descend to f^n on $\mathbb{P}^1$

2) Check that as functions on $\mathbb{P}^1$ they are meromorphic!

Eg: $(z, w) \longmapsto z/w$.

$$R(z, w) = \frac{P(z, w)}{Q(z, w)}$$

2-homogeneous polynomials of same degree

- Meromorphic functions on $\mathbb{P}^1$.

Rewrite this (sort by factorization)

$$R(z, w) = \prod_i (b_i z - a_i w)^{e_i}$$

$$\operatorname{ord}_{[a_i : b_i]}(R) = e_i$$

Thm: Every meromorphic function on $\mathbb{P}^1$ is a ratio of homogeneous polynomials in z, w of the same degree.

$\Sigma e_i = 0$ [... as in the case of Riemann]

$\sum_i e_i = 0$ [same as in the case of sphere]

$\sum \operatorname{ord}_p(R) = 0$.

Torus $\quad X = \mathbb{C}/L$, $\quad L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ upper half plane

$$\pi: \mathbb{C} \to \mathbb{C}/L$$

$f: W \to \mathbb{C}$ be a complex valued function, $W \overset{\text{open}}{\subset} \mathbb{C}/L$

Check. • f is meromorphic at a point $p \in W$ iff there is a preimage z of p in $\mathbb{C}$ such that $f \circ \pi$ is meromorphic at z.

• f is meromorphic on W iff $f \circ \pi$ is meromorphic on $\pi^{-1}(W)$.

• Note that $g = f \circ \pi$ is always L-periodic (doubly periodic), that is $g(z+\omega) = g(z)$ for every $z \in \mathbb{C}$ & every $\omega \in L$, $\omega = n + \tau m$

1-1 correspondence b/w functions on $\mathbb{C}/L$ and L-periodic f^n on $\mathbb{C}$.

Meromorphic L-periodic is called elliptic function

Qus: How do we construct a L-periodic functions?

The analogue of homogeneous polynomials on torus is (Jacobi) theta-function!

Fix a τ with $\operatorname{Im}(\tau) > 0$ and define

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i [n^2\tau + 2nz]}$$

Ex: This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

$\Rightarrow \Theta(z)$ is an analytic function on all of $\mathbb{C}$.

- Note that $\Theta(z+1) = \Theta(z)$, so Θ is periodic for every z in $\mathbb{C}$.

- $\Theta(z+\tau) = e^{-\pi i[\tau + 2z]}\Theta(z)$

(1) + (2) $\Rightarrow$

$$\Theta(z_0) = 0 \text{ iff } \Theta(z_0 + m + n\tau) = 0 \quad \forall\, m, n \in \mathbb{Z}$$

Moreover, the order of zero at z_0 is the same as that at $z_0 + m + n\tau$.

Ex. sheet: The only zeros of Θ are at

$$1/2 + \tau/2 + m + n\tau, \quad m, n \in \mathbb{Z}.$$

and all zeroes are simple.

- Now consider the translates

$$\Theta^{(x)}(z) = \Theta(z - 1/2 - \tau/2 - x)$$

which has simple zeros at the points $x \in L$.

- $\Theta^{(x)}(z+1) = \Theta^{(x)}(z)$
- $\Theta^{(x)}(z+\tau) = -e^{-2\pi i(z-x)}\,\Theta^{(x)}(z)$

Consider the ratio

$$R(z) = \frac{\prod_i \Theta^{(x_i)}(z)}{\prod_j \Theta^{(y_j)}(z)}$$

$R(z)$ is meromorphic on $\mathbb{C}$.

- $R(z+1) = R(z)$
- $$R(z+\tau) = \frac{\prod_{i=1}^{m} \Theta^{(x_i)}(z+\tau)}{\prod_{j=1}^{n} \Theta^{(y_j)}(z+\tau)} = \frac{(-1)^{m-n}\prod_{i=1}^{m} e^{-2\pi i(z-x_i)}\,\Theta^{(x_i)}(z)}{\prod_{i=1}^{n} e^{-2\pi i(z-y_j)}\,\Theta^{(y_j)}(z)}$$

$$(-1)^{m-n}\, e^{-2\pi i[(m-n)z + \sum_j y_j - \sum_i x_i]}\, R(z)$$

We want to have $R(z+\tau) = R(z)$ $\overset{\shortparallel}{1}$

$\Updownarrow$

$m - n = 1$

$\sum_i x_i - \sum_j y_j \in \mathbb{Z}$.

<u>Prop^n</u>: Fix a positive integer d, and choose any 2 sets of d complex numbers $\{x_i\}$ and $\{y_j\}$ such that $\sum_i x_i - \sum_j y_j \in \mathbb{Z}$.
Then the ratio of translated theta functions

$$R(z) = \frac{\prod_i \Theta^{(x_i)}(z)}{\prod_j \Theta^{(y_j)}(z)}$$

is a meromorphic L-periodic function on $\mathbb{C}$ and so descends to a meromorphic function on $\mathbb{C}/L$.

- Zeroes of $R(z) = x_i + L$
  poles of $R(z) = y_j + L$

These are all the meromorphic function on the torus.

<u>Meromorphic functions on smooth plane curves</u>

let $f(x,y) = 0$ define a smooth affine plane curve $X \subseteq \mathbb{C}^2$

Any polynomial $g(x,y)$ is holomorphic on X.

$\Rightarrow \quad r(x,y) = \dfrac{g(x,y)}{h(x,y)}$ is meromorphic if $h(x,y) \not\equiv 0$ on X

$\Updownarrow$

$f(x,y) \nmid h(x,y)$

For projective plane curves

$X = Z(F) \subseteq \mathbb{P}^2$

$\uparrow$ homogeneous

Take ratios $\quad G(x,y,z)$ homogeneous of

$\overline{H(x,y,z)}$ same degree

well def^n everywhere in $\mathbb{P}^2 \setminus Z(H)$

$H \not\equiv 0$ iff $F \nmid H$ on X.

. Meromorphic (check on charts). $\square$

<u>Def^n</u>: Let X be a Riemann surface, which is a subset of a projective space $\mathbb{P}^n$. We say X is <u>holomorphically embedded</u> in $\mathbb{P}^n$ if for every point $p \in X$, there is a homogeneous coordinate z_j such that

a. $z_j \neq 0$ at p.

b. for every k, the ratios z_k/z_j are holomorphic functions on X near p and

c. there is a homogeneous coordinate z_i such that the ratio z_i/z_j is a local coordinate on X near p.

A Riemann surface which is holomorphically embedded in projective space is called a smooth projective curve.

<u>Ex</u>: $\mathbb{P}^1$, smooth projective plane curve $\subseteq \mathbb{P}^2$, any (local) complete intersection are all smooth projective curves.

<u>Pf</u>: Suppose X is a local complete intersection in $\mathbb{P}^n$. Fix $p \in X$. Near p, say in U, X is locally the graph of a set of $n-1$ holomorphic functions of a complex variable z and therefore we can write

$X \cap U$ as the locus

$$[1 : z : g_2(z) : \dots : g_n(z)] \quad \text{(after reording)}$$

Here homogeneous coordinate z_0 is non-zero at p, and the ratio $z = z_1/z_0$ is a local coordinate at p and all the ratios z_i/z_0 are holomorphic at p. This is true for any p

ratios z_k/z_0 ...

$\Rightarrow$ X is holomorphically embedded into $\mathbb{P}^n$.

$\square$.

For a Riemann surface X to be holomorphically embedded in $\mathbb{P}^n$ it is equivalent that X is locally a graph of $n-1$ holomorphic functions.

Indeed, if we fix $p \in X$ and assume that z_0 is the non-zero homogeneous coordinate on X near p, then it is clear that near p, X is the graph of

$$[1 : z : g_2(z) : \dots : g_n(z)]$$

where $g_k(z) = z_k/z_0$ is holomorphic! $\square$

Observation: Meromorphic functions on smooth projective curves.

Ratios of homogeneous polynomials. $R = \dfrac{G(z_0, z_1, \dots, z_n)}{H(z_0, \dots, z_n)}$

with G, H of same degree & as long as $H \not\equiv 0$ on X.

$\square$.

Holomorphic maps b/w R.S:

Defn: • A mapping $F : X \to Y$ is holomorphic at $p \in X$ iff $\exists$ charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$ such that the composition $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$.

• If F is defined on some open $W \subset X$, then we say F is holomorphic on W if F is holomorphic at each point of W.

• In particular, F is holomorphic map iff F is holomorphic on all of W.

Trivial examples: $id : X \to X$

$f : X \to \mathbb{C}$ holomorphic function

If F is holomorphic $\Rightarrow$ F is contn & C^∞.

- If F is holomorphi…

- F, G holomorphic maps $\Rightarrow$ $F \circ G$ holomorphic.

$\Rightarrow$ We can make a category of R.S. with objects R.S

morphisms holomorphic maps b/w them.

holomorphic map

- Composition $F \circ f : Y \to X \to \mathbb{C}$

holomorphic function

meromorphic

then $F \circ f$ is holomorphic fⁿ

~~meromophic fⁿ~~

$(*\ F(Y) \not\subseteq \text{Poles of } f)$

$\square$