# EXERCISE SHEET 5: ALGEBRAIC CURVES AND RIEMANN SURFACES

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## 1. Defining Sheaves

These are gadgets which keep track of both the global and local functions and relations between them at the same time.

Definition 1. Let $X$ be a topological space. A **presheaf** $\mathcal{F}$ of sets (resp. abelian groups, rings, $\mathbb{C}-$algebras, $R$-modules) on $X$ consists of the data

(1) for every open subset $U \subseteq X$, a set $\mathcal{F}(U)$ (resp. abelian group, ring, $\mathbb{C}-$algebra, $R$-module)
(2) for every inclusion $V \subset U$ of open subsets of $X$, a morphism of sets $\rho_{UV}$ : $\mathcal{F}(U) \to \mathcal{F}(V)$ (resp. of abelian groups, rings, $\mathbb{C}-$algebras, $R$-modules), subject to the conditions
  (a) $\mathcal{F}(\emptyset) = \emptyset$, where $\emptyset$ is the empty set, (resp. 0),
  (b) $\rho_{UU}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
  (c) if $W \subset V \subset U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

(1) (1 pt) Rephrase the above definition in terms of categories.
(2) (2 pt) Show that attaching to every open set of a Riemann surface, the set of holomorphic functions on that open set, gives a presheaf.
(3) (1 pt) Show that attaching to every open set of a Riemann surface, the set of meromorphic functions on that open set, gives a presheaf.
(4) (1 pt) Show we can make the above two as presheaf of $\mathbb{C}-$ algebras.

   Definition 2. A presheaf $\mathcal{F}$ on a topological space $X$ is a sheaf if it satisfies the following conditions:
   (a) if $U$ is an open set, if $\{V_i\}$ is an open covering of $U$ and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then s =0;
   (b) if $U$ is an open set, if $\{V_i\}$ is an open covering of $U$ and if we have elements $s_i \in \mathcal{F}(V_i)$ for each $i$, with the property that for each $i, j$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each $i$. Note that the previous condition implies that such an $s$ is unique.

(5) (2 pts) Show that the two presheaves above are sheaves.

## 2. World of Riemann Surfaces-from Miranda's book

(1) (2 pts) Show that the series defining the theta-function converges absolutely and uniformly on compact subsets of $\mathbb{C}$.
(2) (2 pts) Show that the zeros of the theta function $\theta$ are at the points $1/2 + \tau/2 + m + n\tau$, where $m, n \in \mathbb{Z}$ and that these zeroes are simple.

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(3) (2 pts) Let $X$ be a Riemann surface, construct a 1-1 correspondence (with proof) between the set of meromorphic functions on $X$ and the set of holomorphic maps $F : X \to \mathbb{C}_\infty$ which are not identically $\infty$. Give the corresponding map to also $\mathbb{P}^1$.

(4) (6 pts) Recall that a lattice $L \subset \mathbb{C}$ is an additive subgroup generated (over $\mathbb{Z}$) by two complex numbers $\omega_1$ and $\omega_2$ which are linearly independent over $\mathbb{R}$. Thus $L = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$.

(a) Suppose that $L \subset L'$ are two lattices in $\mathbb{C}$. Show that the natural map from $\mathbb{C}/L$ to $\mathbb{C}/L$ is holomorphic and is biholomorphic iff $L = L'$.

(b) Let $L$ be a lattice in $\mathbb{C}$ and let $\alpha$ be a non zero complex number. Show that $\alpha L$ is a lattice in $\mathbb{C}$ and that the map

$$\phi : \mathbb{C}/L \to \mathbb{C}/(\alpha L)$$

sending the coset $z + L$ to $(\alpha z) + \alpha L$ is a well defined biholomorphic map.

(c) Show that every torus $\mathbb{C}/L$ is isomorphic to a torus which has the form $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau$ is a complex number with strictly positive imaginary part.

(5) (2 pts) Let $f$ be any non-constant meromorphic function on a complex torus $X = \mathbb{C}/L$. Then show that $\sum_p ord_p(f) = 0$.

## 3. Orientability

(1) (2 pts) Construct a cover of the Möbius strip by complex charts and write down the transition functions for this cover.

(2) (2 pts) Show that the transition functions you constructed above is not sense preserving, thereby conclude that the Möbius strip is not orientable.