EXERCISE SHEET 3: ALGEBRAIC CURVES AND RIEMANN SURFACES

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DUE DATE: 12 October 2021

1. Exercises on Complex analysis and manifolds

Definition 1. A closed subset of $\mathbb{C}^n$ is called a **closed analytic subspace** if, for every point $x \in X$, there exists an open neighborhood U of x in $\mathbb{C}^n$ and finitely many holomorphic functions $f_1, \dots, f_r$ on U so that

$$X \cap U = \{y \in U | f_i(y) = 0 \ \forall \ 1 \le i \le r\}.$$

Note that any algebraic subset (i.e., zero set of finitely many polynomials)of $\mathbb{C}^n$ (with usual subspace topology not Zariski topology) is automatically a closed analytic subspace.

(1) (2 pts) Give an example of a closed analytic subspace of $\mathbb{C}^n$ which is not an algebraic subspace.
(2) (3 pts) State implicit function theorem for complex multivariate holomorphic functions. Use the statement of the theorem to come up with conditions that will make a closed analytic subspace into a complex manifold.

2. Exercises on Projective varieties

(1) (2pts) Show that an ideal is homogeneous if and oly if it can be generated by homogeneous elements.
(2) (4 pts) Show that sum, product, intersection and radical of homogeneous ideals are homogeneous.
(3) (2 pts) (Homogeneous Nullstellensatz) Prove that if $\mathfrak{a} \subset S = k[x_0, \dots, x_n]$ is a homogeneous ideal and if $f \in S$ is a homogeneous polynomial with degree of $f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.
(4) (3 pts) For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:
   (a) $Z(\mathfrak{a}) = \emptyset$ (the empty set).
   (b) $\sqrt{\mathfrak{a}} = S$ or the ideal is $S_+ = \oplus_{d>0} S_d$.
   (c) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.
(5) (4 pts) Prove that
   (a) There is a 1-1 inclusion reversing correspondence between the algebraic sets in $\mathbb{P}^n$ and homogeneous ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$.
   (b) An algebraic set is irreducible iff $I(Y)$ is a prime ideal.
   (c) $\mathbb{P}^n$ itself is irreducible.
   (d) Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another.
(6) (2 pts) If Y is a projective variety with homogeneous coordinate ring $S(Y)$ show that dim $S(Y) = \dim Y + 1$.

Date: October 5, 2021.

(7) (2 pts) Show that $\dim \mathbb{P}^n = n$.

(8) (6 pts) **Twisted Cubic** Let $Y \subset \mathbb{A}^3$ be the set $Y = (t, t^2, t^3) | t \in k$. Show that Y is an affine variety of dimension 1. Find the generators of the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable. Let $\bar{Y}$ be its projective closure. Find the generators for $I(\bar{Y})$ and use this example to show that if $f_1, \ldots, f_r$ are generator of $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do not necessarily generate $I(\bar{Y})$.