Important Information

Main Reference: Rick Miranda
Algebraic Curves & Riemann Surfaces

Classes: Monday-Tuesday 9:10 - 10:25 am
14 Weeks: September 20, 2021 — December 21, 2021

Tutorial: Everyweek - Friday 5 - 6:15 pm
Office hours: Biweekly - 6:15 - 7:30 pm Fri

Grading Policy: Assignments — 60%
Uploaded on Tue

Quizzes (x2) - 10%

Final Exam - 30%

lecture notes and recordings: Will be available after the lecture on my webpage & Moodle

Link: www.mathface.com/teaching-acrs/
Attn: Recordings will be deleted after one week!

Office Hours: Biweekly - Fridays 6:30 - 7:30

What is a Curve ?

"Every one knows what a curve is, until they have studied enough mathematics to become confused through the countless number of possible exceptions"

—Felix Klein (1958)

What is curve / surface ?

1-dim'l (topological manifold)

Smooth proj. Variety

topological Curve : "X top. space which has dimension 1"

(Krull dimension): If X is a topological space, we define dim X to be the sup of all integers n such that ∃ a chain $Z_0 \subset Z_1 \subset \dots \subset Z_n$ of distinct closed irreducible subsets of X.

What is the dim $\mathbb{R}$?

Only irr. closed subset of $\mathbb{R}$ is pt. singleton.

$\dim \mathbb{R} = 0$!

Not that bad def^n! It's just my topology on $\mathbb{R}$ is too fine!

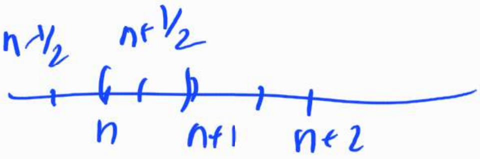
$Z \neq Z' \cup Z''$ ← "union"

$Z', Z'' \subset Z$ "proper closed"

Another def^n: "Lebesgue Covering Dimension"

- A collection $\mathcal{A}$ of subsets of the space X is said to have order $m+1$ if some point of X lies in $m+1$ elements of $\mathcal{A}$ and no point of X lies in more than $m+1$ elements of $\mathcal{A}$.

Eg: $\mathbb{R}$



$\mathcal{A}_0 = \{(n, n+1)\}$

$\mathcal{A}_1 = \{(n-\frac{1}{2}, n+\frac{1}{2})\}$

$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \rightarrow$ order is 2

- A collection $\mathcal{B}$ is said to refine $\mathcal{A}$ or to be a refinement of $\mathcal{A}$ if for each element B of $\mathcal{B}$ $\exists$ an element A of $\mathcal{A}$ such that $B \subset A$.

$\mathcal{B}$: $(n, n+\frac{1}{2})$, $(n-\frac{1}{2}, n)$, $(n-\frac{1}{4}, n+\frac{1}{4})$.

Def^n: A space X is said to be finite dimensional if there is some integer m such that for every open covering $\mathcal{A}$ of X, there is an open covering $\mathcal{B}$ of X that refines $\mathcal{A}$ and has order at most $m+1$.

The top. dim of X is defined to be the smallest value of m for which this statement holds, denoted $\dim X$.

- $\dim \mathbb{R} \leq 1$ [Ex. to show that it is exactly 1].

$\dim [0,1] = 1$

$\dim \mathbb{R}^n = n$.

Caution: (Exercise sheet) $\exists$ a space X (complete metric space) s.t. $\underbrace{\dim X}_{1 =} = \dim X \times X$!

Locally Euclidean space

- A space X is said to be locally m-euclidean if for each $x \in X$, $\exists$ a nbhd of x that is homeomorphic to an open set of $\mathbb{R}^m$.

(Exc. Sheet: Such a space need not be Hausdorff but is a T_1 space)

Moreover, if X is Hausdorff and admits a countable basis, then X is called m-topological manifold.

$$\dim X \times Y = \dim X + \dim Y$$

Complex Structures:

Let X be a topological space

· A complex chart on X is a homeomorphism $\phi : U \to V$, where $U \subset X$ open

domain of ϕ

$V \subset \mathbb{C}$ is an open set in complex plane.

The chart ϕ is said to be centered at $p \in U$ if $\phi(p) = 0$

like giving a complex coordinates on U.

$$z = \phi(x), \quad x \in U.$$

Example A: $X = \mathbb{R}^2$, $U \subset \mathbb{R}^2$ any open set.

$$\phi_U(x,y) = x + iy : U \to \mathbb{C}$$

$$\psi_U(x,y) = \frac{x}{1+\sqrt{x^2+y^2}} + i\frac{y}{1+\sqrt{x^2+y^2}} : U \to \mathbb{C}$$

. Subchart: $\phi : U \to V$ complex chart on X

$U_1 \subset U$ open

Then $\phi|_{U_1} : U_1 \longrightarrow \phi(U_1)$ is a complex chart on X.

- let $\phi : U \to V$ complex chart on X
  $V \subset \mathbb{C}$, $W \subset \mathbb{C}$ s.t. $\psi : V \to W$ holomorphic bijection.

Then

$$\psi \circ \phi : U \to V \to W \text{ complex chart}$$
(Change of coordinates)

<u>Def^n</u>: let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be 2 complex charts on X. We say ϕ_1 is <u>compatible</u> with ϕ_2 if either $U_1 \cap U_2 = \emptyset$ or
$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$
is holomorphic.

- Note def^n is symmetric $(\phi_1 \circ \phi_2^{-1})$

$T = \phi_2 \circ \phi_1^{-1}$ is called Transition function.

Eg Any two sub-charts of a complex chart are compatible.

<u>Ex</u>: In Example A, both charts are not compatible unless the domains are empty / disjoint.

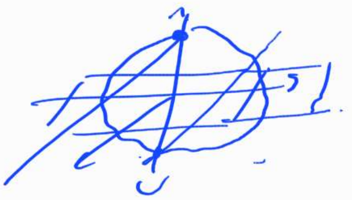
<u>Complex Altases</u>:

<u>Example B</u>: S^2 unit 2-sphere inside $\mathbb{R}^3$
$$S^2 = \{(x, y, \omega) \in \mathbb{R}^3 \mid x^2 + y^2 + \omega^2 = 1\}$$

Identify the plane $\omega = 0$ with $\mathbb{C}$

$$(x, y, 0) \longmapsto z = x + iy.$$

projection:



Let $\phi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$

$$(x, y, \omega) \longmapsto \frac{x}{1-\omega} + i\frac{y}{1-\omega}$$

Complex charts on S^2

$\phi_2 : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$

$$(x, y, \omega) \longmapsto \frac{x}{1+\omega} - i\frac{y}{1+\omega}$$

Ex: • Compute their inverse)

• These charts are compatible.

• Every pt. of the sphere lies in atleast one of the 2 charts.

Def^n: A complex atlas (or simply atlas) on X is a collection $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ of pairwise compatible complex charts whose domains cover X, i.e. $X = \bigcup_\alpha X_\alpha$.

Eg: "Example A, B are atlases."

- It can happen that two different atlases give the same local notion of complex analysis, so we need a notion of equivalence relations on atlases.

Defⁿ: Two complex atlases $\mathcal{A}$ and $\mathcal{B}$ are equivalent if every chart of one is compatible with every chart of the other.

- 2 complex atlases are equivalent iff their union is also a complex atlas.

Ex: Show that every complex atlas is contained in a unique maximal complex atlas.

$\Rightarrow$ Two atlases are equivalent iff they are contained in the same maximal atlas.

Defⁿ: A complex structure on X is a maximal complex atlas on X or equivalently, an equivalence class of complex atlases on X.

Defⁿ: A Riemann Surface is a second countable (i.e admits a countable basis), connected, Hausdorff top. space X together with a complex structure.

Eg: $\mathbb{C}$, S^2 (Riemann sphere).

Remark on defining Riemann Surfaces:

So far, 1. A topo. space
2. A complex atlas on it.

Recipe on how to cook a R.S.

First take top. space & then add complex structure!

This not true:

Observation: If any collection $\{U_\alpha\}$ of subsets of a set X is given and topologies are given for each subset U_α, then one can define a topology on X by declaring a set U to be open iff each intersection $U \cap U_\alpha$ is open in U_α.

Check: This gives a topology, [do you need U_α to cover X?].

Our Recipe for cooking R.S:

1. Start with a set X.
2. Find a countable collection of subsets $\{U_\alpha\}$ of X, which cover X.
3. For each α, find a bijection ϕ_α from U_α to an open set V_α of the complex plane

4. Check that for every α and β,

$$\phi_\alpha(U_\alpha \cap U_\beta) \text{ is open in } V_\alpha.$$

[At this point, we have by above observation a topology defined on X, such that each U_α is open, moreover by def^n each ϕ_α is a complex chart on X.)

5. Check that the complex charts ϕ_α are pairwise compatible

6. Check that X is connected & Hausdorff.

<u>Eg:</u> "The complex projective line"

let $\mathbb{CP}^1$ be the classifying set of 1-dim'l subspaces of $\mathbb{C}^2$ [V.S. dim]

If (z, w) is a non-zero vector in $\mathbb{C}^2$, its span is a point in $\mathbb{CP}^1$, we denote the span of (z, w) by $[z:w]$.

Every pt. of $\mathbb{CP}^1$ can be written as $[z:w]$ with z & w not both zero!

$[z:w] = [\lambda z : \lambda w]$ $\lambda \in \mathbb{C}^*$.

let $U_0 = \{ [z:w] \mid z \neq 0 \}$

$U_1 = \{ [z:w] \mid w \neq 0 \}$

$\phi_0 : U_0 \to \mathbb{C}$ $\qquad$ $\phi_1 : U_1 \to \mathbb{C}$

$[z:w] \mapsto w/z$ $\qquad$ $[z:w] \mapsto z/w$

bijections.

U_0, U_1 cover $\mathbb{C}P^1$.

- $\phi_i(U_0 \cap U_1) = \mathbb{C}^*$ open in $\mathbb{C}$
- charts are compatible. $s \mapsto 1/s$.
- Connected.
- Hausdorff. $\square$.