# EXERCISE SHEET 2: ALGEBRAIC CURVES AND RIEMANN SURFACES

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## 1. Exercises on Riemann Surfaces-From Rick Miranda's book

(1) (2pts) **Projective line is a sphere** Show that the function from $\mathbb{P}^1$ to $S^2 \subset \mathbb{R}^3$ sending
$$[z : w] \mapsto \frac{(2Re(w\bar{z}), 2Im(w\bar{z}), |w|^2 - |z|^2)}{|w|^2 + |z|^2}$$
is a homeomorphism.

(2) (4 pts) Show that the polynomial $f(z, w) = w^2 - h(z)$ is an irreducible polynomial iff $h(z)$ is a polynomial which is not a perfect square. Give the conditions for which $f(z, w)$ is a non singular polynomial and prove it.

(3) (4pts) **Affine Conics** Let $X$ be an affine plane curve of degree 2, that is, defined by a quadratic polynomial $f(z, w)$. Suppose that $f(z, w)$ is singular. Show that $f$ factors as the product of two linear polynomials. Hence, $X$ is a union of two lines. Give an example of smooth affine plane conic.

(4) (2pts) Give an example of a smooth irreducible affine plane curve for a general degree n.

(5) (2pts) Show that any homogenous polynomial satisfies the Euler formula:
$$F = \frac{1}{d} \sum_i x_i \frac{\partial F}{\partial x_i},$$
where $d$ is the degree of $F$.

(6) (6 pts) A degree one projective plane curve, i.e., defined by a linear homogeneous polynomial in $x, y, z$, is called a line. Any such polynomial $F$ is of the form $ax + by + cz$. Write this polynomial in vector form as
$$F(x, y, z) = ax + by + cz = RV = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$
where $R$ is the row vector of coefficients and $V$ is the column vector of variables. Use this description to prove that any two distinct lines in the projective plane meet at a unique point. Give a formula for the point of intersection in terms of the coefficient of lines.

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