EXERCISE SHEET 1: ALGEBRAIC CURVES AND RIEMANN SURFACES

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1. Exercises on Dimension

(1) (2 pts) Find an example of a topological space X such that dim $X \times X$ = dim X, where by dimension we mean the Lebesgue Covering dimension. [Hint: Erdös Space]

(2) (5 pts) Compute the Lebesgue covering dimension for the affine line, $\mathbb{A}^1$.

(3) (10 pts) Explain in your own words: What is a Hausdorff dimension? Why do we need such a notion? Compute it for your favourite space (a fractal or a coastline of a country). Interpret your answer.

2. Exercises on locally eucildean spaces and Riemann Surfaces

(1) (4 pts) **Line with double origin** Consider the following space: Let X be the union of the set $\mathbb{R} - \{0\}$ and the two point set $\{p, q\}$. Declare the topology on X to be given by the basis collection of all open intervals in $\mathbb{R}$ that don't contain 0, and all the sets of the form $(-a, 0) \cup p \cup (0, a)$ and $(-a, 0) \cup q \cup (0, a)$, for $a > 0$.

- (a) (1 pt) Check that the above collection of sets indeed forms a basis for a topology.
- (b) (1 pt) Show that each of the spaces $X - \{p\}$ and $X - \{q\}$ is homeomorphic to $\mathbb{R}$.
- (c) (2 pt) Show that in X all finite point sets are closed but X is not Hausdorff.

(2) (2 pts) Let T be a transition function between two compatible complex charts. Show that the derivative of T is never zero on the domain of T. Show that this implies that we can write T as a power series expansion:

$$z = T(w) = z_0 + \Sigma_{n \geq 1} a_n (w - w_0)^n.$$

3. The World of Algebraic Curves

(1) (2 pts) Identifying $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$, is the Zariski toplogy on $\mathbb{A}^2$ same as the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$?

(2) (Bonus for those attending Algebraic Geometry 1) What is the algebraic version of line with double origin and why is it special?

Date: September 19, 2021.