QUIZ: Algebraic Curves and Moduli

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Rules and Tips

1. You can google or look in any AG book.
2. You can take audience help.
3. You will get 5 min to answer a question.

IMPORTANT TIP

Pay attention to all the questions being asked, you may have to use it to answer the next one. You can keep a note of the questions already answered.

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Round 1

Everybody answers each of the following three questions in chat to me.

Question 1: Can you count?

Question

Is the following functor representable in the category of sets?

$$\begin{aligned} F : \{Sets\}^{op} &\to \{Sets\} \\ X &\mapsto \left\{ \begin{array}{c} \pi : E \to X | \pi \text{ set map} \\ \text{with finite fibers} \\ \text{upto iso.} \end{array} \right\}. \end{aligned}$$

Isomorphism of set families is being an isomorphism bijective on the fibers.

Answer 1: Of course, you can!

Answer

The functor is representable by $\mathbb{N}$, the set of natural numbers. What is the universal family?

Question 2: Hilbert Polynomial does not know this but Hilbert series does?

Question

An example of a family where Hilbert series varies but Hilbert polynomial does not is:

Answer 2

Consider the family $\mathfrak{X} = P \sqcup R \sqcup Q \subset \mathbb{P}^2 \times \mathrm{Spec}(k[t])$, where

$$P : \{x = y + z = 0\}$$

$$Q : \{x = y - z = 0\}$$

$$R : \{y = x - tz = 0\}$$

Fibers	Hilbert Polynomial	Hilbert Series at n=1
Generic fiber $\mathfrak{X}_{(0)}$	3	2
Special fiber $\mathfrak{X}_{(\lambda)}$	3	3

Answer 2: Non-collinear to colinear points

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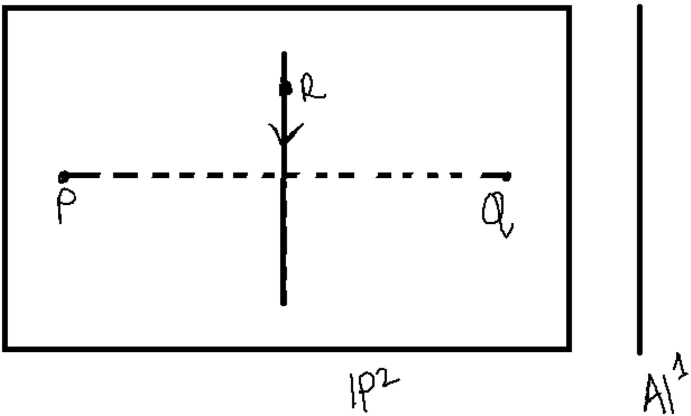

Question 3: Hilbert polynomial still does not get it or does it?

Question

Which subschemes of $\mathbb{P}^3$ have $2n + 2$ as Hilbert polynomial?

Question 3: Hilbert polynomial still does not get it or does it?

Hint to Question

Which subschemes of $\mathbb{P}^3$ have $2n + 2$ as Hilbert polynomial?

Hint: Give atleast two.

Answer

Two skew lines

As they are disjoint, their Hilbert polynomials add up ($h_{line} = n + 1$).

Smooth conic with an embedded point

Hilbert polynomial of a smooth conic is $2n + 1$ with a point raises the constant by 1.
E.g: $\{XY = Z = 0\} \cup \{X = Y = Z^2 = 0\}$

Smooth conic with a point not on the conic

$h_{\text{smooth conic}} = 2n + 1$ and then a disjoint point raises the polynomial by 1.

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Round 2

Each one of you get an individual question, time to give the answer is 5 min. Your time starts after I finish reading the question!

Question 4: One more thing about Hilbert polynomial

Question

What is the Hilbert polynomial of $\mathbb{P}^1$?

Question 4: One more thing about Hilbert polynomial

Question

What is the Hilbert polynomial of $\mathbb{P}^1$ considered as a subscheme of itself and considered as a subscheme of P^d via the Veronese embedding?

Veronese Embedding:

$$v_d : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$

$$[X : Y] \mapsto [X^d : X^{d-1}Y : \ldots : Y^d].$$

Answer: The Hilbert polynomial depends on the embedding

Answer

Hilbert polynomial is $m+1$, $dm+1$ respectively.

Computations: In the first case $\Gamma(\mathbb{P}^1) = k[x,y]$ and $\Gamma(\mathbb{P}^1)_m = Span\{x^m, x^{m-1}y, \dots y^m\}$, so

$$h_{\mathbb{P}^1 \subset \mathbb{P}^1}(m) = \dim_k \Gamma_m(\mathbb{P}^1) = m+1.$$

In the second case, $\Gamma(v_d(\mathbb{P}^1))_m = k[x,y]_{dm}$, thus we have

$$h_{\mathbb{P}^1 \subset \mathbb{P}^d}(m) = \dim_k \Gamma_m(v_d(\mathbb{P}^1)) = md+1$$

Remark about various Degrees

Remark

Let X be a smooth curve and let D be a very ample divisor on X, corresponding to a closed immersion $\varphi : X \hookrightarrow \mathbb{P}^n$, then

$$\deg \varphi(X) = \deg D.$$

Question 5: Relation between $l(D)$ and $\deg D$

Question

Let X be a smooth curve of genus g and D be an effective divisor on X. True or False: $\deg D \geq l(D)$.
Give a proof or an counterexample.

Answer: Relation between $l(D)$ and $\deg D$

Answer

False: For the zero divisor D, $l(D) \neq 0$ but $\deg D = 0$. The relation is $l(D) \leq degD + 1$. And equality occurs if and only if $g = 0$ or $D = 0$.

Proof.

R-R : $l(D) - l(K - D) = \deg D + 1 - g$
In case D is effective we have $l(K - D) \leq l(K) = g$. Thus

$$l(D) - g \leq l(D) - l(K - D) = \deg D + 1 - g.$$

In case $g = 0$, we have $l(K - D) = 0$, hence the equality holds. For the converse, note that $l(K) = l(K - D)$ implies $D \sim 0$ or $g = 0$. □

Question 6: Curves of Genus 2

Question

Give a curve of genus 2 in $\mathbb{P}^2$? Is it smooth? Can you write me a smooth curve of genus 2 in $\mathbb{P}^2$?

Answer: A genus 2 smooth curve cannot be embedded in $\mathbb{P}^2$

Answer

For example: $y^2 = x^5 - 1$, but is not smooth in $\mathbb{P}^2$.
There is **no smooth curve of genus 2** in $\mathbb{P}^2$: as for a smooth genus 2 curve to be embedded in any $\mathbb{P}^n$, you need a very ample line bundle $\mathcal{L}$ which is generated by $n + 1$ global sections, but every very ample line bundle on a smooth genus 2 curve has degree at least 5.(Prove this!)
Then R-R implies that $\mathcal{L}$ is generated by $\deg \mathcal{L} - 1$ global sections, thus the least n we get is 3.

Question 7: Curve of degree d

Question

Let X be a smooth curve of degree d in $\mathbb{P}^n$, with $d \leq n$, which is not contained in $\mathbb{P}^{n-1}$. What is its genus? Does the equality hold between n and d?

Answer: Rational normal curve of degree d

Answer

Its genus is 0 and d=n.
Since $X \subset \mathbb{P}^n$ but not in $\mathbb{P}^{n-1}$, there is a very ample divisor on X of $\deg D = d$ and $l(D) = n + 1$, now take any linearly equivalent effective divisor, since $l(D) \neq 0$, there exists one such.
Then use Question 5, to get $l(D) \leq \deg D + 1$, which gives

$$n + 1 \leq d + 1 \leq n + 1.$$

Thus $d = n$ and the equality holds for $l(D)$ and $\deg D + 1$, which implies that $g = 0$ as $D \not\sim 0$.

Remark

Note that such an curve differs from a Rational normal curve of degree d only by an automorphism of $\mathbb{P}^n$. Rational Normal curve of degree d in $\mathbb{P}^d$ is just the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$.

Question 8: Curve of degree 2 and 3

Question

I have a smooth curve of degree 2 in some $\mathbb{P}^n$, which familiar curve can it look like and in where?

I have a smooth curve of degree 3 in some $\mathbb{P}^n$, which familiar curve(s) can it look like and in where?

Answer: A Curve of degree d is a familiar curve of degree d in $\mathbb{P}^d$

Answer: degree 2

From Question 7, we see that, a curve of degree 2 can be embedded into $\mathbb{P}^2$ such that it is not contained in $\mathbb{P}^1$ but not in any higher dimensional $\mathbb{P}^n$, for $n > 2$ and such a curve has genus 0.
Thus, every degree 2 curve in any $\mathbb{P}^n$ is a conic in $\mathbb{P}^2$.

Answer: degree 3

For degree 3: There are two cases: Either is it contained in $\mathbb{P}^2$ or not, in which case it has to be contained in $\mathbb{P}^3$: In case it is in $\mathbb{P}^2$: it is plane cubic curve: genus 1— Elliptic curve
Or it is a twisted cubic: genus 0 curve in $\mathbb{P}^3$.

Hey, what about embedding in $\mathbb{P}^1$? This is not possible !!

Question 9

Question

Let X be a smooth curve in $\mathbb{P}^3$ but $X \not\subset \mathbb{P}^2$, any plane and $\deg(X \subset \mathbb{P}^3) = d$, genus is g.
Project $\mathbb{P}^3$ from $O \notin X$ to $\mathbb{P}^2$ and let $\varphi(X)$ be the image of X in $\mathbb{P}^2$.
Is $\varphi(X)$ smooth? Why or why not?

Hint: Compute $H^0(X, \mathcal{O}_X(1))$.

Answer

Answer

It is not smooth because assuming otherwise and let us compute $H^1(X, \mathcal{O}_X(1))$.

Then as φ is a non-constant map between smooth curves, it is an isomorphism.

Since $X \subset \mathbb{P}^3 but \not\subset H$, for any hyperplane H, we have $H^0(\mathbb{P}^3, I_X(1)) = 0$.

Thus, $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathcal{O}_X(1))$ is injective, so $h^0(\mathcal{O}_X(1)) \geq 4$.

However, $\varphi(X) \subset in \mathbb{P}^2$ implies that $h^0(\varphi(X), \mathcal{O}_{\varphi(X)}) \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 3$. A contradiction!

Bonus Question: End of Round 2

Question

Let X be a smooth curve in $\mathbb{P}^3$ but $X \not\subset \mathbb{P}^2$, any plane and $\deg(X \subset \mathbb{P}^3) = d$, genus is g. True or False?

$$g < 1/2(d-1)(d-2)$$

Why?

Answer

Answer

True.

Note that X is a normalization of $\varphi(X)$ so we have $g(X) \leq g(\varphi(X)) = 1/2(d-1)(d-2)$ and as $\varphi(X)$ must be not smooth, we have strict inequality.

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Round 3

Now we reverse the order.

Question 10: Curves of degree 4

Question

Let X be a smooth curve of degree 4 in some $\mathbb{P}^n$, what are all the possible genus of such curves?

Answer: Genus of curves of degree 4

Answer

1. g=0, when $X \subset \mathbb{P}^4$ but not in any lower dimensional subspace, so it is a rational normal curve of degree 4 or $X \subset \mathbb{P}^3$ but not in any plane in $\mathbb{P}^3$, rational quartic curve in $\mathbb{P}^3$.
2. $g = 1$ when $X \subset \mathbb{P}^3$
3. $g = 3$, when $X \subset \mathbb{P}^2$: quartic plane curve.

Note that the case $g = 2$ is not possible as for any genus 2, the minimum degree of very ample line bundle is 5.

Question 11: Curves of genus atleast 2

Question

Given a smooth curve of genus atleast two, give a base point free linear system on it? Or equivalently give a globally generated line bundle over it?

Answer: canonical linear system

Answer

$\omega_{X/k}$, the canonical bundle or linear system $|K|$. Another one is given by the divisor $2g[P]$, where P is a point on the curve and g is the genus of the curve.

Question 12: A very ample canonical linear system

Question

Let X be a smooth curve of genus atleast 2, under what condition on X is $\omega_{X/k}$ a very ample line bundle?

Answer

Answer

When X is not hyperelliptic !

This gives the canonical embedding of curves...

Question 13: Hilbert scheme

Question

Which familiar variety is the Hilbert scheme of n points on $\mathbb{P}^1$, i.e., $Hilb^n_{\mathbb{P}^1}$?

Answer

Answer

$\mathbb{P}^n$.

Question 14

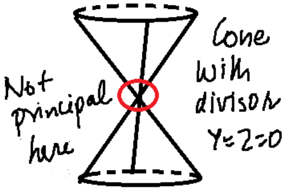
Question: Cartier vs Weil Divisor

Give an example of a Weil divisor which is not Cartier.

Answer: Informally, a Cartier divisor is simply a Weil divisor defined locally by one equation

Answer

Consider the cone: Spec $k[x, y, z]/(xy - z^2)$, and the Weil divisor on it given by $y = z = 0$, this divisor is not Cartier, as the ideal generated by y, z is not a principal ideal.



Question 15: Non trivial Automorphism

Question

Consider the curve

$$x^3y + y^3z + z^3x = 0 \subset \mathbb{P}^2$$

Give a non-trivial automorphism of this curve. (char $\neq 3$)

Answer: There are 167 non-trivial automorphisms of this curve.

Answer

Let ζ_7 be a seventh root of unity, then a non-trivial automorphism is

$$\begin{aligned} x &\mapsto \zeta_7 x \\ y &\mapsto \zeta_7^4 y \\ z &\mapsto \zeta_7^2 z. \end{aligned}$$

The following curve has no automorphisms:

$$y^3 - 3y = x^{g+1} + x^g + 1$$

char $= p \neq 3$, $g \neq (2 \mod 3)$, $g \equiv 0$ or $-1(\mod p)$.

Bonus Question: ~~LADY~~ GAGA

Question

What is the analytification of the scheme $x^2 = 0$ (double line) in $\mathbb{A}^2_{\mathbb{C}}$?

Answer: Not reduced analytic space

Answer

It is the ringed space $(\mathbb{C}, \mathcal{H}/ < z_1^2 >)$, where $\mathcal{H}$ is the sheaf of holomorphic functions on $\mathbb{C}^2$.

End. Thank you!