Lecture 25: Coarse Moduli spaces

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A moduli functor is not rep^n! What to do?

- Ask for less : Coarse moduli space $F: (Sch/\mathbb{C})^{opp} \to (Sets)$

$F \xrightarrow{\simeq} h_X$ (Fine moduli space) $\leftarrow$ unique upto iso.

1) Just ask for a natural transformation

$\Psi_M : F \longrightarrow h_M$ for some $M \in (Sch/\mathbb{C})$

rather than an iso.

$\varphi: \mathcal{D} \to B \in F(B)$ s.t. $\Psi_M(\varphi): B \to M$ { natural with respect to

$\varphi': \mathcal{D}' = \mathcal{D} \times_{B'} B \longrightarrow B'$ $\xi: B' \to B$ }

$\Psi_M(\varphi') = \Psi(\varphi) \circ \xi$.

- Far ~~from~~ determining M.,

$\Psi: F \to h_M$ and take any morphism $\pi: M \to M'$

$\Psi': F \to h_{M'}$ $\Psi' = \pi \circ \Psi$.

$M' = Spec(\mathbb{C})$ and $\Psi(\varphi) = $ structure morphism $h_{Spec(\mathbb{C})}$

$\varphi: \mathcal{D} \to B$ $\downarrow$ $Spec(\mathbb{C})$ $\qquad B \to Spec(\mathbb{C})$

2) We ask that the complex points of M correspond $\overset{1:1}{\longleftrightarrow}$ objects of our moduli problem in $\mathbb{C}$.

$\to$ The scheme structure on M is not fixed.

Eg: Moduli functor $(\mathbb{C})$ of lines through the origin in $\mathbb{C}^2$. "Grassmanian"

we cannot rule out $\mathbb{P}^1$ instead taking cuspidal curve

$\mathbb{P}^1 \longrightarrow M' : y^2z = x^3$ in $\mathbb{P}^2$

$[a,b] \qquad [a^2b, a^3, b^3]$.

3) M is universal with respect to the existence of the natural transformation Ψ:

Def^n: A scheme M and a natural transformation Ψ_M from the functor F to h_M is a coarse moduli space if

1) The map $\Psi_{Spec(\mathbb{C})} : F(Spec\,\mathbb{C}) \longrightarrow M(\mathbb{C}) = Mor(Spec(\mathbb{C}), M)$ set bijection

2) Given another scheme M' & $\Psi': F \to h_{M'}$, there is a unique morphism $\pi: M \to M'$ s.t., the associated nat. transf. $\Pi: h_M \to h_{M'}$ satisfies

$$\Psi_{M'} = \Pi \circ \Psi_M.$$

Ex: (1) Show, coarse moduli space, if it exists, for F, is unique upto an canonical isomorphism.

(2) Cuspidal curve is not a coarse moduli space for line in $\mathbb{C}^2$.

(3) Fine moduli space is a coarse moduli space

(4) j-line is a coarse moduli space for M_1.

Def^n: In case F admits a coarse moduli space M, we define a tautological family over M to be a family X/M s.t for each closed point $m \in M$, the fiber X_m is the element of $F(\mathbb{C})$ corresponding to m by the set bijection of $F(\mathbb{C})$ corresponding to m by the set bijection

$$M(\mathbb{C}) \to h_M(\mathbb{C}).$$

Ex: Show that j-line does not admit a tautological family either.

Example: Consider the moduli problem

$$F : \{Sch/\mathbb{C}\}^{op} \longrightarrow \{sets\}$$

$$S \longmapsto \left\{ \begin{array}{l} \text{set of flat families of 'reduced'} \\ \text{plane curves of degree 2 upto iso} \end{array} \right\}$$

$$F(\mathbb{C}) = \left\{ \left[\begin{array}{c}\text{smooth}\\ \text{conic}\end{array}\right], \left[\begin{array}{c}\text{pair of}\\ \text{distinct line}\end{array}\right] \right\}$$

There is a trivial natural transformation

$$\Psi : F \longrightarrow h_{spec(\mathbb{C})}$$

S - $\mathbb{C}$-scheme $\downarrow_{\mathbb{C}}$ $\quad \mathcal{X} \downarrow S$ $\quad S \to \mathbb{C}$ structure morphism.

Now fix any pair (X, Ψ') where X is a scheme, $\Psi' : F \to h_X$.

$\varphi : \mathcal{C} \to B$ any family of smooth conics, then $\exists$ a unique $\mathbb{C}$-valued point $\pi : Spec(\mathbb{C}) \to X$ s.t. $\Psi'(\varphi) = \pi \circ \Psi(\varphi)$

$B \to X \qquad B \to \mathbb{C} \xrightarrow{\pi} X$

$$\varphi \in F(B) \xrightarrow{\Psi'} h_X(B)$$

$$\Psi \downarrow \quad h_{Spec(\mathbb{C})}(B)$$

$\Psi(\varphi) =$ str morphism

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\alpha"] \arrow[dr, "id"'] & B \arrow[d, "surj"] & \leftarrow Y \\ & \mathbb{C} & \end{tikzcd}$$

$h_B(\mathbb{C}) \to h_{\mathbb{C}}(\mathbb{C})$ surj $\quad$ id

$$\begin{tikzcd} \mathcal{C}_{\mathbb{C}} \arrow[r] \arrow[d, "\varphi_{\mathbb{C}}"'] & \mathcal{C} \arrow[d] \\ \mathbb{C} \arrow[r, "\alpha"] \arrow[dr, "id"'] & B \arrow[d, "str"] \\ & \mathbb{C} \end{tikzcd}$$

$$\begin{array}{c} B \\ \downarrow \text{surj.} \\ \mathbb{C} \end{array} \qquad \begin{array}{c} h_B(\mathbb{C}) \\ \downarrow \text{surj} \\ id \in h_{\mathbb{C}}(\mathbb{C}) \end{array}$$

$\varphi_{\alpha'} = \varphi_{\alpha} \in F(\mathbb{C})$ $\Psi'(\varphi_{\mathbb{C}}) \in X(\mathbb{C})$

π $Spec(\mathbb{C}) \to X$

Claim: π is unique does not depend on choice of 'α'.

Ex: Let $\varphi : \mathcal{C} \to \mathbb{A}^1_t$ be the family defined by the (affine) eq^n $xy - t$ and φ' its restriction to $\mathbb{A}^1 \setminus \{0\}$.

- φ' is a family of smooth conics

Show that $\Psi'(\varphi) = \pi \circ \Psi(\varphi)$ for the unique π as above.

Note that the pair $(spec(\mathbb{C}), \Psi)$ has the universal property 2).

- But 1) is not satisfied

Conclude that F admits no coarse moduli space. $\square$.

$$M_g : \{Sch/\mathbb{C}\}^{opp} \longrightarrow \{Sets\}$$

$$S \longmapsto \left\{ \begin{array}{l} \text{smooth families with geom. fiber. projective} \\ \text{curves of genus } g \ /S \end{array} \right\}$$

$\mathcal{M}_0$: genus 0 curves.

$\mathcal{M}_0(\mathbb{C}) = \{pt.\}$

<u>Prop^n</u>: $M = \operatorname{Spec}(\mathbb{C})$ is a coarse moduli space for $\mathcal{M}_0$ and it has a tautological family: $(\mathbb{P}^1/\mathbb{C})$

Coarse moduli space: i) ✓ obvious

ii) $\Psi : \mathcal{M}_0 \to h_{\operatorname{Spec}(\mathbb{C})}$ structure morphism.

Suppose $\Psi' : \mathcal{M}_0 \to h_N$ is another morphism.

$e : \mathbb{C} \to N$ $\qquad \Psi'_{\mathbb{C}} : \mathcal{M}_0(\mathbb{C}) \to h_N(\mathbb{C})$.

We need to show that Ψ' factors through the morphism $\varphi : F \to h_M$ described above.

Let X/S family of curves of genus 0, S scheme of finite type.

$\forall s \in S$ closed $X_s \cong \mathbb{P}^1$, so every point s must go to the same point $n_0 \in N$ as the image of the morphism $e : \mathbb{C} \to N$.

We need to show that morphism $S \to N$ factor through the reduced point n_0 as a closed subscheme of N.

[Deformation Theory] The restriction of the family on S to any artinian closed subscheme of S will be <u>trivial</u>, and therefore will factor through the reduced scheme $\operatorname{Spec}(k)$.

$\mathbb{P}^1$ is rigid under deformation. $\Leftrightarrow$ $H^1(T) = 0$

Similar argument works for schemes of not finite type; make base ext^n to geom. points of S and consider Artin rings over them.

∘ $\mathbb{P}^1/\mathbb{C}$ is the tautological family.

<u>Claim</u>: The moduli space is not fine.

$\Rightarrow$ The moduli functor $\mathcal{M}_0$ is not representable.

Why?: Think about ruled surfaces: there exist ruled surfaces which are not globally trivial:

- <u>Def^n</u>: A ruled surface is a surface X, together with a surjective morphism $\pi : X \to C$ to a non-singular curve C, such that the fibers $X_y \cong \mathbb{P}^1$ for each $y \in C$ and π admits a section.

curve' C, such that the fibers $X_y \cong \mathbb{P}^1$ for each $y \in C$ and π admits a section.

These closed surfaces are locally trivial [locally $U \times \mathbb{P}^1$ $U \subseteq C$ open.
but in general $X \not\cong C \times \mathbb{P}^1$.]

Ex: Find one example of such a surface.

Eg: Families of curves of genus 0 need not be even locally trivial.

$A = k[t,u]$, consider the curve $k = \bar{k}$
in $\mathbb{P}^2_A$ defined by $tx^2 + uy^2 + z^2 = 0$.

Take $S = \operatorname{Spec} A \setminus \{tu = 0\}$ X family of curves over S.

This family is not even locally trivial.

Generic fiber X_η defined over $K = k(t,u)$
has no rational points [check] □.

Def^n: An n-pointed ~~smooth~~ smooth curve $(C, p_1, \dots, p_n)$ is a projective smooth curve C equipped with a choice of n-distinct marked points $p_1, \dots, p_n \in C$.

$g = 0$, $C \cong \mathbb{P}^1$

Iso: $\varphi : (C, p_1, \dots, p_n) \xrightarrow{\sim} (C', p_1', \dots, p_n')$
is an iso $\varphi : C \xrightarrow{\sim} C'$
$\varphi(p_i) = p_i'$ $i = 1, \dots, n$.
[order is preserved]

More generally. a family of n-pointed smooth rational curves is a flat and proper $\pi : \mathcal{X} \to B$ with n disjoint sections $\sigma_i : B \to \mathcal{X}$ such that each geom. fiber $\mathcal{X}_b = \pi^{-1}(b)$ is a proj. smooth rational curve

Note that the n-sections single out n special points $\sigma_i(b)$ which are the n-marked points of that fiber.

An iso between two families

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\sim"] \arrow[d, "\pi"'] & \mathcal{X}' \arrow[d, "\pi'"'] \\ B \arrow[u, bend right, "\sigma_i"'] \arrow[r, equal] & B \arrow[u, bend right, "\sigma_i'"'] \end{tikzcd}$$

φ : & for each i.

<u>Thm</u>: If $\mathcal{X} \to B$ is a flat family with geom. fibers iso to $\mathbb{P}^1$ that admits at least one section, then $\mathcal{X} \cong \mathbb{P}(E)$ for some rank 2 vector bundle E on B.

If the family admits at least two disjoint sections, then the bundle splits;

and if there are at least three disjoint sections then $\mathcal{X} \cong B \times \mathbb{P}^1$.

and there is a unique morphism such that the three sections are identified with the constant sections $B \times \{0\}$, $B \times \{1\}$, $B \times \{\infty\}$ in this order.

Thus if $n \geq 3$, for any given family $\mathcal{X} \to B$ of n-pointed curves (smooth) curves there is a unique B-isomorphism $\mathcal{X} \xrightarrow{\sim} B \times \mathbb{P}^1$, so that all the information of the family is in the section.

classifying them is same as classifying n-tuples of distinct points in a fixed $\mathbb{P}^1$, upto <u>projective equivalence</u>.

$\mathrm{Aut}(\mathbb{P}^1) \ni \sigma : (p_i) \longmapsto (p_i')$

$M_{0,3}$: is a fine moduli space given by Spec $\mathbb{C}$

$g^{\prime\prime}, \pi$. $\cdot (C, p_1, p_2, p_3) \underset{\text{uniquely}}{\cong} (\mathbb{P}^1, 0, 1, \infty)$.

- All family are trivial., pull them back from

$M_{0,4} := \underline{\mathbb{P}^1 \setminus \{0,1,\infty\}}$

<u>Cross-ratio</u> : $(p_1, p_2, p_3, p_4) \in \mathbb{P}^1$

$\exists!$ automorphism $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$

s.t.
$$\begin{aligned} p_1 &\mapsto 0 \\ p_2 &\mapsto 1 \\ p_3 &\mapsto \infty. \\ p_4 &\mapsto \sigma(p_4) \in \mathbb{P}^1 \setminus \{0,1,\infty\} \end{aligned}$$

$$\begin{tikzcd} U_{0,4} : \mathbb{P}^1 \setminus \{0,1,\infty\} \times \mathbb{P}^1 \arrow[d] \\ \mathbb{P}^1 \setminus \{0,1,\infty\} \arrow[u, bend left, "z_1"] \arrow[u, bend left=40, "z_2"] \arrow[u, bend left=60, "z_3"] \arrow[u, bend left=80, "z_4"] \end{tikzcd}$$

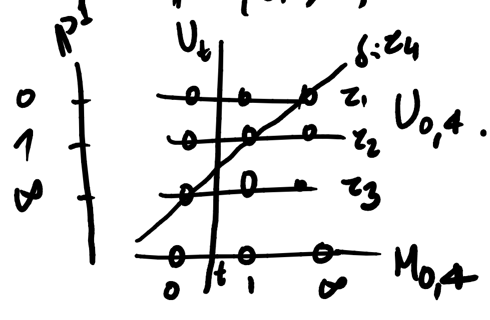
z_1 : constant section $\{0\} \times \mathbb{P}^1 \setminus \{0,1,\infty\}$

z_2 : —— $\{1\} \times$ —

z_3 : ——. $\{\infty\} \times$ —

z_4 : diagonal section

$$\delta : \mathbb{P}^1 \setminus \{0,1,\infty\} \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{0,1,\infty\}$$
$$\mathbb{P}^1 \subset \qquad\qquad \mathbb{P}^1 \times \mathbb{P}^1 \subset$$



<u>Check</u>: Universal property of this family.

<u>Construction</u> of M_g as a coarse moduli space via GIT.

<u>Idea</u>: For any integer $n \geq 3$, any curve C we can embed C as a curve of degree $2(g-1)n$ in projective space $\mathbb{P}^N = \mathbb{P}^{(2n-1)(2g-1)-1}$ by the complete linear system $|nK_C|$

Consider pairs $(C, \varphi_n : C \to \mathbb{P}^N)$ ← we know how to parametrize

Consider pairs $(C, \varphi_n : C \to \mathbb{P}^N)$ ← we know how to parametrize them.

K locus parametrizing such curves is a locally closed subset K of the Hilbert scheme $\mathcal{H} = \mathcal{H}_{\underline{(2g-1)n, g}, N}$.

φ_n depends on a choice of a basis for the space $H^0(C, K_C^{\otimes n})$ of n-canonical differential on C

Such a choice ⟷ group $PGL(N+1, \mathbb{C})$ acts on K.

and this quotient if it exists, should be M_g.
GIT allows you to construct it. □