Lecture 23: Tangent Space of Hilbert Scheme

Saturday, May 30, 2020 6:03 PM

$H := \mathrm{Hilb}^{P(t)}_{\mathbb{P}^r}$ $\quad \Sigma = \mathrm{Spec}\, \mathbb{C}[\varepsilon]/\varepsilon^2$

$$T_h H \xleftrightarrow[\text{(as a set)}]{1:1} \mathrm{Hom}\left(\left(\mathbb{C}[\varepsilon]/\varepsilon^2, \varepsilon_0\right), (H, h)\right) = \left\{ \text{set of flat families } \begin{array}{c} \mathbb{P}^r \times \Sigma \supset X \\ \downarrow f \\ \Sigma \end{array} \text{ s.t. fiber over closed pt. of } \Sigma \text{ is } X_0 \subseteq \mathbb{P}^r \right\}$$

More generally given a scheme Y, $X_0 \subseteq Y$ closed, first order deformation as embedded scheme of X_0 is just a flat family

$$\begin{array}{c} X \subset Y \times \Sigma \\ f \downarrow \swarrow \\ \Sigma \end{array}$$

whose fiber over closed point is $X_0 \subset Y$.

$Y = \mathrm{Spec}\, R$, R-$\mathbb{C}$-algebra

$R[\varepsilon]/\varepsilon^2 = R \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]/\varepsilon^2$ is $Y \times \underbrace{\mathrm{Spec}\, \mathbb{C}[\varepsilon]/\varepsilon^2}_{\Sigma}$

$X \subset Y \times \Sigma$ closed corresponds to an ideal $J \subset R[\varepsilon]/\varepsilon^2$

$X \xrightarrow{f} \Sigma$ flat $\Rightarrow$ $\left(R[\varepsilon]/\varepsilon^2\right)/J$ is flat as $\mathbb{C}[\varepsilon]/\varepsilon^2$ module.

Claim: flatness of f $\Leftrightarrow$ every presentation by gen. & relations of the ideal I of the closed fiber $\underline{X_0} = \underline{X \bmod (\varepsilon)}$ extends to a presentation of J.

Lemma 5.1: Let $\varphi: A \to B$ be a hom. of noetherian comm. rings, B flat $/A$. Assume either A is local Artinian ring or A, B are local rings and φ local homo.

$J \subset B$ ideal, $C = B/J$

$k = A/m_A$. $B_k = B \otimes_A k$, $C_k = C \otimes_A k$.

TFAE:

(i) C is flat over A (5.2)

(ii) every exact sequence $\underbrace{B_k^{\ell} \to B_k^{h}}_{I} \to B_k \to C_k \to 0$ is reduction modulo m_A of an exact seq^n $\underbrace{B^{\ell} \to B^{h}}_{J} \to B \to C \to 0$.

$\Downarrow$ obvious

(iii) there are generators $F_1, \dots, F_h$ of J such that denoting by the f_i the image of F_i in B_k, $i = 1, \dots, h$. every relation among f_i extends to a relation among the F_i.

We will use following facts:

Lemma 5.3 R com. ring if SES R-modules $0 \to E \to F \to G \to 0$ and G is flat, then E is flat $\Leftrightarrow$ F is flat $\forall H$ R-module

Pf: $\dots \to \mathrm{Tor}_2^R(G, H) \to \mathrm{Tor}_1^R(E, H) \xrightarrow{\subset} \mathrm{Tor}_1^R(F, H) \to \mathrm{Tor}_1^R(G, H) \to$

$\mathrm{Tor}_2^R(G,H) = 0$ (G-flat), $\mathrm{Tor}_1^R(E,H) = 0$, $\mathrm{Tor}_1^R(F,H) = 0$, $\mathrm{Tor}_1^R(G,H) = 0$.

Lemma 5.4: (Local criterion for flatness) A f.g. B-module (or any B-module if A is artinian) M is flat over A iff $\mathrm{Tor}(A/m, M) = 0$.

Lemma 5.4. [illegible]
B-module if A is artinian) M is flat over A iff
$$\mathrm{Tor}_1^A(M,k)=0.$$

Pf: $\boxed{\Rightarrow}$ obvious

$\boxed{\Leftarrow}$ A Artinian local.

$\mathrm{Tor}_1^A(M,N)=0$ for any f.g. A-module N

A Artinian $\Rightarrow \exists\, \nu \in \mathbb{Z}$ s.t $m_A^\nu = 0$,

it suffices to show that $\mathrm{Tor}_1^A(M,N_i)=0$ for each quotient $N_i = m_A^i N / m_A^{i+1} N$ of the composition series
$$N = m_A^0 N \supset m_A^1 N \supset \dots \supset m_A^\nu N = 0.$$
and just note that $\mathrm{Tor}_1^A(M,N_i)=0$ as N_i is a finite dim'l k-v.s. $\square$.

Pf of lemma 5.1:

i) $\Rightarrow$ ii) Write $C_k = B_k/I$, I ideal of B_k

Since $C_k = B/(J + m_A B) \Rightarrow I \cong J/(J\cap m_A B)$

Assume C is flat over A + $B_k^l \to B_k^h \to B_k \to C_k \to 0$

Apply Lemma 5.3 to $\quad 0 \to J \to B \to C \to 0 \quad (*)$

$\Rightarrow J$ is flat $/A$.

Now we tensor $(*)$ with k, we get
$$\underbrace{\mathrm{Tor}_1^A(C,k)}_{0 \text{ flatness of } C} \to J\otimes_A k \to B_k \to C_k \to 0$$
$\Rightarrow J\otimes_A k = I \iff m_A J = J\cap m_A B$
as $J\otimes_A k = J/m_A J$

We can construct a commutative diagram

$$\begin{tikzcd} B_k^h \arrow[r, "\beta"] & I \arrow[r] & 0 \\ B^h \arrow[u] \arrow[r, "\alpha"] & J \arrow[u] \arrow[r] & Q \end{tikzcd}$$

Claim: α is onto.

$Q\otimes_A k = \mathrm{coker}\,\beta = 0$. $\Rightarrow Q = m_A Q$

If B is local ring $\Rightarrow Q = m_B Q \Rightarrow Q = 0$ by Nakayama's lemma

If A is Artinian. $\exists\, \nu\in\mathbb{N}$ s.t $m_A^\nu = 0$
$$Q = m_A Q = m_A^2 Q = \dots = m_A^\nu Q = 0 \Rightarrow Q = 0.$$
$\Rightarrow$ we can extend generators of I to J.

It remains to show we can extend relations:

let $N = \ker\beta$, $M = \ker\alpha$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & B_k^h & \longrightarrow & I & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M & \longrightarrow & B^h & \longrightarrow & J & \longrightarrow & 0 \end{array} \quad \boxed{} \leftarrow \otimes k.$$

$\underset{\text{''}}{M\otimes_A k}$ + use J is flat $\Rightarrow M \otimes_A k = N$.

$$\begin{array}{ccccc} B_k^l & \longrightarrow & N & \longrightarrow & 0. \\ \uparrow & & \uparrow & & \\ B^l & \xrightarrow{\gamma} & M & & \end{array}$$

Argue as before. $\Rightarrow \gamma$ is onto, so we have

$B^l \to B^h \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ is an exact sequence extending (5.2)

$B_k^l \to B_k^h \to B_k \to C_k \to 0$

(iii) $\Rightarrow$ (i) Choice of generators $F_1, \dots, F_h$ corresponds to a surj. morphism

$0 \to M \to B^h \twoheadrightarrow J$

$f_1, \dots, f_h \quad \rightsquigarrow 0 \to N \to B_k^h \twoheadrightarrow I$

Cond iii) $\Rightarrow M \twoheadrightarrow N$

Thus if $B^l \to M$ is onto $\Rightarrow B_k^l \twoheadrightarrow N$

$B^l \to B^h \to B \to C \to 0 \quad (**) \quad \otimes k$

we get exact. $B_k^l \to B_k^h \to B_k \to C_k \to 0$

We can complete $(**)$ to a free resolution of C.

tensor the resolution with k, we obtain a complex whose homology calculates $\mathrm{Tor}_i^A(C, k)$

$(**)$ complex is exact in degree 1 $\Rightarrow \mathrm{Tor}_1^A(C, k) = 0$

lemma 5.4 $\Rightarrow C$ flat$/A$. $\square$

Corollary 5.7: Let A & B be as above. Let $f_1, \dots, f_h$ be elements of $B/m_A B$ and for each $i = 1, \dots, h$, let F_i be the elements in B which reduce to f_i mod $m_A B$. Then $B/(F_1, \dots, F_h)$ is flat$/A$ iff every relation among f_i extends to a relation among F_i.

Lemma 5.8: R comm. Noetherian $\mathbb{C}$-algebra and let I be an ideal R.

Lemma 5.8: R comm. Noetherian $\mathbb{C}$-algebra and let I be an ideal R.

The first order embedded deformations of $X_0 = \operatorname{Spec}(R/I)$ within $Y = \operatorname{Spec}(R)$ are in 1-1 correspondence with

$$\operatorname{Hom}_{R/I}(I/I^2, R/I) = \operatorname{Hom}_R(I, R/I)$$

Pf: $g \in R$, $[g] := g \bmod I$

We have to classify ideals $J \subset R[\varepsilon]/\varepsilon^2$ s.t. $R[\varepsilon]/\varepsilon^2/J$ flat over $\mathbb{C}[\varepsilon]/\varepsilon^2$

and $J/_{(\varepsilon)\cap J} = I$

Assume we are given such a J.

Given $i \in I$, pick $j \in J$ s.t. $i = j \bmod (\varepsilon)$; now we can write $j = i - \varepsilon h$, $h \in R$ for some.

- h depends R-linearly on j
- [h is determined uniquely by i mod I.]

Indeed, if $i = 0$, then $\varepsilon h \in J \cap (\varepsilon)$

flatness of $R[\varepsilon]/\varepsilon^2/J \Rightarrow J \cap (\varepsilon) = \varepsilon J = \varepsilon I$

$\Rightarrow h \in I$.

This gives us a map
$$I \longrightarrow R/I$$
$$i \longmapsto [h]$$

Conversely suppose we are given a homomorphism $\alpha : I \to R/I$ R-linear

Choose generators $f_1, \ldots, f_n$ for I. Write

$\alpha(f_i) = [g_i]$ where $g_i \in R$ and

set $F_i = f_i - \varepsilon g_i$, $J = (F_1, \ldots, F_n)$.

Clearly, $J/(J \cap (\varepsilon)) = I$.

Claim: $R[\varepsilon]/\varepsilon^2/J$ is flat over $\mathbb{C}[\varepsilon]/\varepsilon^2$

By corollary above all we have to show: if

$\sum a_i f_i = 0$, $a_i \in R$ is a relation among f_i, it comes from a relation among F_i mod (ε)

$\check{}$ $\sum a_i f_i = 0$, $a_i \in R$ is a relation among f_i, it comes from a relation among f_i mod (ε).

$$\sum a_i [g_i] = \alpha\left(\sum a_i f_i\right) = 0$$

$\Rightarrow$ $\sum a_i g_i \in I$, so $\underline{\sum a_i g_i = \sum b_i f_i}$ for some elements $b_i \in R$

Thus, $\sum (a_i + \varepsilon b_i) F_i = \underbrace{\sum a_i f_i}_{0} + \varepsilon(\underbrace{\sum b_i f_i - \sum a_i g_i}) = 0$.

is relation along F_i which extends $\sum a_i f_i = 0$

Check: The two maps above are inverse to each other. $\square$.

Let $X \subset Y$ closed subscheme, I ideal of X in Y

Defn: Conormal sheaf: Sheaf $I/I^2 |_X$

denoted $\mathcal{C}_{X/Y}$

Its dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_Y}(I, \mathcal{O}_X)$ is the normal sheaf of X in Y and is denoted by $N_{X/Y}$.

Propn: 5.9 Let $X \subset Y$ closed subscheme with ideal I. Then the first order embedded deformations of X in Y are in 1-1 correspondence with $H^0(X, N_{X/Y}) = \mathrm{Hom}(\mathcal{C}_{X/Y}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(I, \mathcal{O}_X)$.

Corollary: The tangent space to $\mathrm{Hilb}_r^{P(H)}$ at h is given by
$$T_h(\mathrm{Hilb}_r^{P(H)}) = H^0(X, N_{X/\mathbb{P}^r})$$

In particular, we have that $h^0(X, N_{X/\mathbb{P}^r})$ is an <u>upperbound</u> for the dimension of $\mathrm{Hilb}_r^{P(t)}$ at h.

<u>Lower bound</u>:

Propn: Let X be a closed lci subscheme of $\mathbb{P}^r$ and let h be the corresponding point of $\mathrm{Hilb}_r^{P(t)}$. Then the dimension of every irreducible component of $\mathrm{Hilb}_r^{P(t)}$ at h is atleast $h^0(X, N_{X/\mathbb{P}^r}) - \underbrace{h^1(X, N_{X/\mathbb{P}^r})}_{\text{"obstruction to deformation"}}$.

Let $\mathcal{X} \subset Y \times B$

obstruction to deformation.

let $X \subset Y \times B$ be a flat family of subschemes parametrized by a scheme B

$$\begin{array}{c} X \subset Y\times B \\ \downarrow \\ B \end{array}$$

$X = X_{b_0}$ for some closed point $b_0 \in B$.

Let v be a tangent vector to B at b_0. $\Leftrightarrow$

$$v : \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \longrightarrow (B, b_0)$$

Pull back $\mathcal{X}$ via v, giving you a first order deformation of X in Y.

$\Leftrightarrow$ giving an element of $H^0(X, N_{X/Y})$.

So we have a map $T_{b_0}(B) \longrightarrow H^0(X, N_{X/Y})$. (5.10).

Kodaira-Spencer map / Charateristic map.

[Read: Arabello et al. Geometry of curves. chapter 9. §6]

Ex:A : Show that this is a linear map.

Example :- H be the Hilbert scheme of degree d zero dim'l subschemes of $\mathbb{P}^r$

and consider a point of H $\leftrightarrow$ $Z \subset \mathbb{P}^r$

If Z consists of d distinct points $p_1, \dots, p_d$, then

$$H^0(Z, N_{Z/\mathbb{P}^r}) = \bigoplus_{i=1}^{d} T_{p_i}(\mathbb{P}^r) = T_{p_1+\dots+p_d}(\operatorname{Sym}^d(\mathbb{P}^r))$$

$\Rightarrow$ the open subset of $\operatorname{Sym}^d(\mathbb{P}^r)$ consisting of d-uples of distinct points embedds in H as an open subset.

- Consider hypersurfaces of degree d in $\mathbb{P}^r$.

$$\operatorname{Hilb}_r^{\text{degree } d \text{ hyp.}} \cong \mathbb{P}^N \qquad N = \binom{d+r}{r} - 1$$

$$h^0(X, N_{X/\mathbb{P}^r}) = h^0(X, \mathcal{O}_X(d)) = \binom{d+r}{r} - 1$$

Qn: What would make a moduli functor to be not rep^n?...