Lecture 21: Construction of Hilbert scheme: Effective Serre's Theorem

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Fix $\mathbb{P}^r$, $p(t) \in \mathbb{Q}[t]$

$$\mathrm{Hilb}_r^{p(t)} \subset G = G(q(n), H^0(\mathbb{P}^r, \mathcal{O}(n)))$$

$$q(n) = h^0(\mathbb{P}^r, \mathcal{O}(n)) - p(n) = \binom{n+r}{r} - p(n)$$

$$X \in \mathbb{P}^r,\ p(t) \longmapsto \underbrace{H^0(\mathbb{P}^r, \mathcal{I}_X(n))}\subset H^0(\mathbb{P}^r, \mathcal{O}(n))$$

$\dim h^0(\mathbb{P}^r, \mathcal{I}_X(n)) = q(n)$ $n \gg 0$.

Lemma: let $r \in \mathbb{Z}_{\geq 0}$ and $q(t) \in \mathbb{Q}[t]$, then $\exists$ an integer n_0 s.t. for any ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r}$ with Hilbert polynomial $q(t)$ and for any $n \geq n_0$

(i) $H^i(\mathbb{P}^r, \mathcal{I}(n)) = 0$ for every $i \geq 1$

(ii) the natural map
$$H^0(\mathbb{P}^r, \mathcal{I}(n)) \otimes H^0(\mathbb{P}^r, \mathcal{O}(1)) \longrightarrow H^0(\mathbb{P}^r, \mathcal{I}(n+1))$$
is surjective.

Pf: By induction on r.

For $r = 0$, ntp

$r > 0$, X projective scheme defined by $\mathcal{I}$.

Choose a hyperplane H not containing any of the components of X, including embedded points.

$$\Rightarrow H \cap \underbrace{\mathrm{Ass}(X)}_{\text{finite}} = \Phi$$

Set $\mathcal{J} = \mathcal{I} \otimes \mathcal{O}_H$.

[$0 \to \mathcal{O}_{\mathbb{P}^r}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_H \to 0 \quad \otimes \mathcal{O}_X$

$$0 = \mathrm{Tor}_1(\mathcal{O}_{\mathbb{P}^r}^{\text{flat}}, \mathcal{O}_X) \to \mathrm{Tor}_1(\mathcal{O}_H, \mathcal{O}_X) \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X \to \dots$$

by choice of H α is injective, so $\mathrm{Tor}_1(\mathcal{O}_H, \mathcal{O}_X) = 0$.]

Tensoring $0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_X \to 0 \quad \otimes \mathcal{O}_H$

$$\Rightarrow \boxed{0 \to \mathcal{I} \otimes \mathcal{O}_H = \mathcal{J} \to \mathcal{O}_H \to \mathcal{O}_X \otimes \mathcal{O}_H \to 0.}$$

$\mathrm{Tor}_1(\mathcal{O}_H, \mathcal{O}_X)$. $\quad 0 \to \mathcal{J} \to \mathcal{O}_H \to \mathcal{O}_X \otimes \mathcal{O}_H \to 0$ (A.2)

Pf: $i: X \hookrightarrow \mathbb{P}^n$, $j: H \hookrightarrow \mathbb{P}^n$ degree d. hypersurface.

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to j_* \mathcal{O}_H \to 0 \quad (1)$$

$H \cap \mathrm{Ass}(X) = \phi \iff (1) \otimes \mathcal{O}_X$ is left exact.

We check exactness locally pick any affine open $A \subset \mathbb{P}^n$

f degree d polynomial in A. $I \subset A$ for X.

$$0 \to A \xrightarrow{\times f} A \to A/f \to 0 \quad \otimes A/I$$

$$0 \to A/I \xrightarrow{\times f} A/I$$
$$\Updownarrow$$
$$\mathrm{Ass}(A/I) \cap \mathrm{Spec}(A/(f)) = \phi$$

$$\Updownarrow$$
$$\mathrm{Ass}(A/I) \cap \mathrm{Spec}(A/(f)) = \emptyset$$

- Let M be an A-module and $f \in A$. Then f is M-regular iff $\mathrm{Ass}(M) \cap \mathrm{Spec}(A/(f)) = \emptyset$.

$\boxed{\Rightarrow}$ f is M-regular $\Rightarrow$ f is not a zero division of M

$\Rightarrow$ $\mathrm{Ass}(M) \cap \mathrm{Spec}(A/(f)) = 0$.

prime ideals of the form $\mathrm{Ann}(x)$ $x \in M \setminus \{0\}$ — prime ideals containing f

$\boxed{\Leftarrow}$ f is not M-regular $\Rightarrow$ $\exists f \in \mathrm{Ann}(m)$ for some $m \neq 0$.

$\{\mathrm{Ann}(n) \mid \mathrm{Ann}(n) \supset \mathrm{Ann}\, m\}$

this set has a maximal element — prime ideal.

We have $f \in \mathrm{Ann}(m) \subset \mathfrak{p}$ i.e. $\mathrm{Ass}(M) \cap \mathrm{Spec}(A/f) \neq \emptyset$. $\square$

$$0 \to \mathcal{O}_{\mathbb{P}^r}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_H \to 0 . \quad \otimes \mathcal{I}(m+1).$$

one gets an exact sequence:

$$\boxed{0 \to \mathcal{I}(m) \to \mathcal{I}(m+1) \to \mathcal{J}(m+1) \to 0} \quad (A\cdot 3)$$

$\leftarrow$ Why is this exact?

$$\begin{array}{ccccc} 0 \to & A & \xrightarrow{f} & A \\ & \cup & & \cup \\ 0 \to & I & \xrightarrow{f} & I \end{array}$$

Thus, the Hilbert polynomial of $\mathcal{J}$ satisfies.

$$h_{\mathcal{J}}(t) = h_I(t) - h_I(t-1) = q(t) - q(t-1)$$

$\therefore$ only depends on $q(t)$ and not on I and H.

By induction, $\exists\, n_1$ such that i) and ii) are satisfied for $\mathcal{J}$ whenever $n \geq n_1$.

Use (4.3) $\Rightarrow$ $H^i(\mathbb{P}^r, \mathcal{I}(n)) \cong H^i(\mathbb{P}^r, \mathcal{I}(n+1))$ $i \geq 2$

$[H^i(\mathbb{P}^r, \mathcal{J}(n)) = 0 \;\; \forall n > n_1]$

As $H^i(\mathbb{P}^r, \mathcal{I}(m)) = 0$ $\forall\, m \gg 0$

$\Rightarrow$ $H^i(\mathbb{P}^r, \mathcal{I}(n)) = 0$ for $i \geq 2$.

$\Rightarrow H^i(\mathbb{P}^r, I(n)) = 0$ for $i \geq 2$.

We are left to prove the case $H^1(\mathbb{P}^r, I(n)) = 0$.

For $n \geq n_1$, there is an exact sequence

$$H^0(\mathbb{P}^r, I(n+1)) \xrightarrow{\alpha_n} H^0(\mathbb{P}^r, \mathcal{J}(n+1)) \to H^1(\mathbb{P}^r, I(n)) \to H^1(\mathbb{P}^r, I(n+1)) \to 0$$

Either a) α_n is surjective

b) $h^1(\mathbb{P}^r, I(n+1)) < h^1(\mathbb{P}^r, I(n))$

Observe: If α_n is surjective $\Rightarrow$ α_{n+1} is surjective.

Pf:

$$H^0(\mathbb{P}^r, I(n+1)) \otimes H^0(\mathbb{P}^r, \mathcal{O}(1)) \to H^0(H, \mathcal{J}(n+1)) \otimes H^0(H, \mathcal{O}(1))$$

surjective as α_n is surj

$\downarrow$ $\leftarrow$ surjective by induction

$H^0(H, \mathcal{J}(n+2))$

$\Rightarrow$ Image in $H^0(\mathbb{P}^r, I(n+2))$ of $H^0(\mathbb{P}^r, I(n+1)) \otimes H^0(\mathbb{P}^r, \mathcal{O}(1))$ already surjects into $H^0(H, \mathcal{J}(n+2))$. $\square$

In conclusion

$n \to \infty$ $\quad h^1(\mathbb{P}^r, I(n)) \longrightarrow 0$,

as $h^1(\mathbb{P}^r, I(n)) = 0 \quad \forall n \gg 0$.

$\Rightarrow H^1(\mathbb{P}^r, I(n)) = 0$ if $n \geq n_1 + \underbrace{h^1(\mathbb{P}^r, I(n_1))}$

$\exists$ upper bound for $h^1(\mathbb{P}^r, I(n_1))$ independent of I.

$$h^1(\mathbb{P}^r, I(n_1)) = h^0(\mathbb{P}^r, I(n_1)) - q(n_1) \leq h^0(\mathbb{P}^r, \mathcal{O}(n_1)) - q(n_1).$$

Claim: $n_0 = n_1 + h^0(\mathbb{P}^r, \mathcal{O}(n_1)) - q(n_1) + 1$ will do!

i) ✓ $[n \geq n_0 - 1]$

ii) Consider the diagram $\quad a \qquad \alpha(a)$ exact.

$\ldots H^0(\mathbb{P}^r, I(n+1)) \xrightarrow{\alpha} H^0(\mathbb{P}^r, \mathcal{J}(n+1))$

ii) Consider the diagram

$$\begin{tikzcd} H^0(\mathbb{P}^r, \mathcal{I}(n)) \arrow[r, "\eta"] & H^0(\mathbb{P}^r, \mathcal{I}(n+1)) \arrow[r, "\alpha"] & H^0(\mathbb{P}^{r}, \mathcal{J}(n+1)) \\ & H^0(\mathbb{P}^r, \mathcal{I}(n)) \otimes H^0(\mathbb{P}^r, \mathcal{O}(1)) \arrow[u, "\beta"] \arrow[r, "\gamma"] & H^0(\mathbb{P}^r, \mathcal{J}(n)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}}(1)) \arrow[u, "\delta"] \end{tikzcd}$$

a $\quad \alpha(a)$ exact.

we have to show this is surjective $\to \beta$

$\gamma(c) = b$, $\delta(b) = \alpha(a)$

δ surjective by induction hypo $n_0 > n_2$

γ surjective for $n \geq n_0$.

im $\eta \subset$ im β
$\Rightarrow \beta$ is onto. $\quad$ as i) holds for any $n \geq n_0 - 1$. $\square$

Remark: Also holds in case we replace $\mathcal{I}$ with a coherent subsheaf of a fixed coherent sheaf on $\mathbb{P}^r$.
[Used in Quot schemes]

Corollary 4.5: Let $r \in \mathbb{Z}_{>0}$, $p(t) \in \mathbb{Q}[t]$. Then $\exists n_0 \in \mathbb{Z}$ with the following property:

Let $X \subset \mathbb{P}^r \times S$ be any flat family of subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, $\psi: X \to S$

$\mathcal{I}_X$ ideal sheaf of X

Then for any $n \geq n_0$, the following holds

i) $\psi_* \mathcal{I}_X(n)$ is locally free of rank $q(n) = \binom{n+r}{r} - p(n)$

ii) $R^i \psi_* \mathcal{I}_X(n) = 0 \quad i \geq 1$

(iii) multiplication map $\psi_* \mathcal{I}_X(n) \otimes \psi_* \mathcal{O}_{\mathbb{P}^r \times S}(1) \to \psi_* \mathcal{I}_X(n+1)$ is onto

(iv) for any morphism $\alpha: T \to S$, the natural homomorphism $\alpha^* \psi_* \mathcal{I}_X(n) \xrightarrow{\sim} \varphi_* \mathcal{I}_Y(n)$ is an iso, where $Y = X \times_S T \subset \mathbb{P}^r \times T$ and $\varphi: \mathbb{P}^r \times T \to T$ proj.

Pf: I_X is flat over S as for $m \gg 0$

$$0 \to \psi_* I_X(m) \to \underline{\psi_* \mathcal{O}_{\mathbb{P}^r \times S}(m)} \to \underline{\psi_* \mathcal{O}_X(m)} \to 0$$

is exact

locally free as X is flat over S.

$\Rightarrow \psi_* I_X(m)$ is locally free

$\Rightarrow I_X$ is flat.

let $n_0 \in \mathbb{Z}$ be as in lemma above + base change (iv) & higher direct maps.

for ii) + iii)

Ex.

$\square$

<u>Back to construction of hilbert scheme</u>:

For any $n \geq n_0$: (from corollary above)

$X \subset \mathbb{P}^r \xrightarrow{\text{associate}}$ n-th Hilbert point i.e. as $n \geq n_0$

$\varphi_n : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(X, \mathcal{O}_X(n))$

$H^0(\mathbb{P}^r, I_X(n))^{\cup}$

point in $G = G(q(n), H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)))$.

n-th Hilbert point determines X completely:

$\Longleftrightarrow$ homogeneous ideal of X is generated in degree n or more by $H^0(\mathbb{P}^r, I_X(n))$.

i) $\Rightarrow \dim H^0(\mathbb{P}^r, I_X(m)) = q(m)$ for $m \geq n_0$.

ii) $\Rightarrow$ n th Hilbert point of $X \in H \subseteq G$, where H contains points of G that are vector sub-spaces $V \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$ s.t image of

$$f_{m,V} : V \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-n)) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$$

$\| \qquad\qquad\qquad\qquad \subseteq H^0(\mathbb{P}^r, I_X(m))$

$H^0(\mathbb{P}^r, I_X(n)) \qquad\qquad\qquad\qquad q(m)$.

has dimension $q(m)$

Conversely suppose $V \in H$, Then just by def^n V generates a homogeneous ideal, $h_{I(V)}(t) = q(t)$

it comes from closed subscheme of $\mathbb{P}^r$ with hilbert polynomial $p(t)$.

1-1 correspondence

$\{X \subseteq \mathbb{P}^r_k \mid h_X(t) = p(t)\} \longleftrightarrow H$

$X \longmapsto n$-th hilbert point