Lecture 18: More on Flatness

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Solutions to exercise sheet 2 online!

Propⁿ: Let X be a noetherian scheme and let $f: X \to Y$ be a morphism, $Y = \operatorname{Spec} A$. Then for any quasicoherent $\mathcal{F}$ on X, we have

$$R^i f_*(\mathcal{F}) \cong \widetilde{H^i(X, \mathcal{F})} \quad [\text{sheaf associated to } H^i(X,\mathcal{F})]$$

on Y.

Proof: Recall $f_*\mathcal{F}$ is a quasicoherent sheaf on Y.

Hence $f_*\mathcal{F} \cong \widetilde{\Gamma(Y, f_*\mathcal{F})}$

But $\Gamma(Y, f_*\mathcal{F}) = \Gamma(X, \mathcal{F})$, so we have an iso $i=0$.

$\sim$: is an exact functor $\operatorname{Mod}(A)$ to $\operatorname{Mod}(Y)$.

$R^i f_*$, $\tilde{H^i}$ both δ-functors from $\operatorname{Qco}(X)$ to $\operatorname{Mod}(Y)$.

$$\mathcal{F} \hookrightarrow \mathcal{G} \quad (\text{flasque qcoh}) \text{ on } X.$$

[Pf: Cover X with finite # open affines $U_i = \operatorname{Spec} A_i$

$$\mathcal{F}|_{U_i} = \tilde{M_i} \text{ for each } i.$$

$M_i \hookrightarrow I_i$ (A_i-module injective)

For each i, let $f: U_i \hookrightarrow X$ inclusion $\mathcal{G} = \bigoplus f_*(\tilde{I_i})$

For each i $\mathcal{F}|_{U_i} \to \tilde{I_i}$ injective.

$$\mathcal{F} \longrightarrow f_*(\tilde{I_i})$$

Take direct sum over I $\mathcal{F} \hookrightarrow \mathcal{G}$.]

$\Rightarrow$ Both the δ-functor $R^i f_*$ $\tilde{H}^i(X, .)$ are effaceable for $i > 0$: we conclude there is a unique iso reducing to given one for $i=0$. $\square$.

Corollary: $f: X \to Y$, X noetherian, $\mathcal{F} \in \operatorname{Qcoh}(X)$, then the sheaves $R^i f_*(\mathcal{F})$ are qcoh on Y.

Pf: local result: use Propⁿ above.

Propⁿ: Let $f: X \to Y$ morphism of separated noetherian schemes. Let $\mathcal{F} \in \operatorname{Qcoh}(X)$, $\mathcal{U} = (U_i)$ affine open cover of X, $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ Čech resolution of $\mathcal{F}$. Then $p \geq 0$

$$R^p f_*(\mathcal{F}) \cong h^p(f_* \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})).$$

Proof: Exercise.

Thm 8.8: Let $f: X \to Y$ be a projective morphism of noetherian

Thm 8.8: Let $f: X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{O}_X(1)$ be a very ample sheaf on X/Y, $\mathcal{F} \in Coh(X)$. Then:

a) $\forall\, n \gg 0$, the natural map $f^* f_*(\mathcal{F}(n)) \to \mathcal{F}(n)$ is surjective.

b) $\forall\, i \geq 0$ $R^i f_*(\mathcal{F})$ is a coherent sheaf on Y;

c) for $i > 0$ and $n \gg 0$, $R^i f_*(\mathcal{F}(n)) = 0$.

Proof. We can be local on Y, $Y = \operatorname{Spec} A$.

Then using Prop^n 8.5 above, (a) says that $\mathcal{F}(n)$ is generated by global sections; Serre's result [Hart II.5.17]

b) $\iff$ $H^i(X, \mathcal{F})$ is finitely generated A module (Serre's result on cohomo. projective III 5.2a)

c) $\iff$ $H^i(X, \mathcal{F}(n)) = 0$ $\forall\, n \gg 0$ (III 5.2b) $\square$

Flatness with higher direct image ... (Theory of base change)

Setup:

$$\begin{tikzcd} g^*\mathcal{F} \quad \mathbb{P}^r \times T \arrow[r, "g'' = id \times f"] \arrow[d, "\eta"] & \mathbb{P}^r \times S \quad (\mathcal{F}) \arrow[d, "\xi"] \\ R^i \eta_* g^* \mathcal{F} \quad T \arrow[r, "f"] & S \quad R^i \xi_* \mathcal{F} \end{tikzcd}$$

$\mathcal{F}$ coherent sheaf on $\mathbb{P}^r \times S$

$f^* R^i \xi_* \mathcal{F}$

Assumption: $\mathcal{F}$ is flat over S!

Theory of base change: [Ref^2: Mumford Abelian Varieties §5 Theorem]

Any point of S has an open nbhd $U \subset S$ over which $\exists$ a bounded complex
$$K^0 \to K^1 \to K^2 \to \cdots$$
of locally free coherent sheaves which calculates the direct images of $\mathcal{F}$ functorially under base change.

Thus, for any base change $f: T \to S$ s.t. $f(T) \subset U$ the cohomology of the complex
$$f^* K^0 \to f^* K^1 \to f^* K^2 \to \cdots$$
is $R^i \eta_* g^* \mathcal{F}$.

$$\begin{tikzcd} g^*\mathcal{F} \quad \mathbb{P}^r \times T \arrow[r, "g"] \arrow[d, "\eta"] & \mathbb{P}^r \times S \quad \mathcal{F} \arrow[d, "\xi"] \\ R^i \eta_* g^* \mathcal{F} \quad T \arrow[r, "f"] & S \quad R^i \xi_* \mathcal{F} \end{tikzcd}$$

$f^* R^i \xi_* \mathcal{F}$

Prop^n: TFAE:

(i) $R^q \xi_* \mathcal{F} = 0$ for $q > 0$

Propⁿ: TFAE:

$R^1\eta_* g^*\mathcal{F} \quad ? \quad \mapsto S\, R^1\xi_*\mathcal{F}$
$f^* R^1 \xi_* \mathcal{F}$

(i) $R^q \xi_* \mathcal{F} = 0$ for $q > 0$

(ii) $R^q \eta_* g^* \mathcal{F} = 0$ for any base change $T \xrightarrow{f} S$ and any $q > 0$

(iii) $H^q(\mathbb{P}^r, \mathcal{F}_s) = 0$ for any $s \in S$ and any $q > 0$

(iv) ______________ for any closed point s of S and any $q > 0$.

Moreover, if one of the above holds, $\xi_* \mathcal{F}$ is locally free, and the natural homomorphisms

$$f^* \xi_* \mathcal{F} \longrightarrow \eta_* g^* \mathcal{F}$$
$$\xi_* \mathcal{F} \otimes k(s) \longrightarrow H^0(\mathbb{P}^r, \mathcal{F}_s)$$

are isomorphisms for <u>any base change</u> and any $s \in S$.

Proof: (i) ⇒ (ii) $K^\cdot$ is a locally free resolution of $\xi_* \mathcal{F}$ ⇒ $\xi_* \mathcal{F}$ is locally free

(1) $0 \to \xi_* \mathcal{F} \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^n \to 0$
exact sequence

Ex: $\psi : \mathcal{F} \to \mathcal{G}$ surjective morphism of locally free sheaves then $\ker \psi$ is locally free!

⇒ (1) is exact sequence of locally free sheaves and hence of flat $\mathcal{O}_S$-modules.

⇒ $0 \to f^* \xi_* \mathcal{F} \to f^* K^0 \to f^* K^1 \to \cdots$ is also exact (Exercise)

Thus $R^q \eta_* g^* \mathcal{F} = 0$ for $q > 0$ and $\eta_* g^* \mathcal{F} \cong f^* \xi_* \mathcal{F}$.

(iv) ⇒ (i) The complex
$$K^0 \otimes k(s) \longrightarrow K^1 \otimes k(s) \longrightarrow K^2 \otimes k(s) \longrightarrow \cdots$$
is exact for any closed s

Claim: $K^0 \longrightarrow K^1 \longrightarrow \cdots$ is exact as well.

We use induction on the length of complex
$$L^0 \longrightarrow L^1 \longrightarrow \cdots \longrightarrow L^n \longrightarrow 0$$
of locally free coherent sheaves on U
is exact if $L^\cdot \otimes k(s)$ is exact for any $s \in S$.

$n=0$: obvious

For $n>0$ $\quad L^{n-1}\otimes k(s) \to L^n \otimes k(s) \to 0$ for any s

Nakayama's lemma $\Rightarrow L^{n-1} \to L^n \to 0$

$\Rightarrow$ Kernel of this morphism, Z^{n-1} is locally free and $Z^{n-1}\otimes k(s) =$ the kernel of $L^{n-1}\otimes k(s) \to L^n \otimes k(s) \to 0$

$\Rightarrow L^0 \to \dots \to Z^{n-1} \to 0 \quad (2)$

tensor with $k(s)$ yield an exact complex for any closed point s. By induction hypothesis $\Rightarrow$ (2) is exact. $\square$

Prop: let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S \times S$ and denote by ξ the projection of $\mathbb{P}^n \times S$ onto S. Then $\mathcal{F}$ is flat over S iff $\xi_*(\mathcal{F}(n))$ is locally free for $n \gg 0$.

Proof: Suppose $\mathcal{F}$ is flat over S.

If $n \gg 0$, then $R^i \xi_* \mathcal{F}(n) = 0$

hence, the prop^n above tells us that $\xi_*(\mathcal{F}(n))$ is locally free.

Converse, Assume $\xi_*(\mathcal{F}(n))$ is locally free $\forall n \gg 0$.

To show $\mathcal{F}$ is flat$/S$ we show that for any injection $\mathcal{G}_1 \hookrightarrow \mathcal{G}_2$ of coherent $\mathcal{O}_S$-modules

$\xi^* \mathcal{G}_1 \otimes \mathcal{F} \hookrightarrow \xi^* \mathcal{G}_2 \otimes \mathcal{F}$ is an injection.

wlog $S = \operatorname{Spec} A$.

let $R = \bigoplus_{n \geq 0} R_n$ homogeneous coordinate ring for $\mathbb{P}^n_S$, $R_0 = A$

$\mathcal{G}_i = \widetilde{G_i}$ $\quad G_i$ f.g. A-modules

$\mathcal{F} = \widetilde{F}$ $\quad F = \bigoplus_{n \geq n_0} F_n$ f.g. graded R-module.

$\xi_*(\mathcal{F}(n))$ locally free $n \gg 0 \iff F_n$ is a projective A-module $n \gg 0$.

Assume F_n is projective and hence flat over A $\forall n \geq n_0$

Assume F_n is projective and hence flat over A
$\forall n \geq n_0$

Then it follows that $\xi^* \mathcal{G}_1 \otimes \mathcal{F} \hookrightarrow \xi^* \mathcal{G}_2 \otimes \mathcal{F}$ as

$$\xi^* \mathcal{G}_i \otimes \mathcal{F} = \widetilde{(G_i \otimes_A R) \otimes_R F}$$
$$= \widetilde{G_i \otimes_A F}$$

and flatness of F over A. $\square$

Back to Hilbert schemes:

Claim: $\mathbb{P}^N$ is the Hilbert scheme of hypersurfaces of degree d in $\mathbb{P}^r$ and that $\pi : \mathcal{X} \to \mathbb{P}^N$ is its universal family.

$$N = \binom{d+r}{r} - 1$$

$$\begin{tikzcd}
X = \alpha^*\mathcal{X} \arrow[r] \arrow[dr, phantom, "\square"] \arrow[d] & \mathcal{X} := \sum_I a_I x^I = 0 \subseteq \mathbb{P}^r \times \mathbb{P}^N \arrow[d, "\pi"] \\
S \arrow[r, "\alpha"] & \mathbb{P}^N
\end{tikzcd}$$

$x_0, \ldots, x_r$ homogeneous coordinates for $\mathbb{P}^r$

a_I homo $\mathbb{P}^N$ [I runs through multindices $(i_0, \ldots, i_r)$ s.t. $\sum i_k = d$]

Proof: let $X \subset \mathbb{P}^r \times S$ be a flat family of hypersurfaces of degree d in $\mathbb{P}^r$, with $f : X \to S$ and $p_2 : \mathbb{P}^r \times S \to S$.

We want to show that this family comes from a morphism $\alpha : S \to \mathbb{P}^N$

Now let I be the ideal sheaf of X in $\mathbb{P}^r \times S$

$$0 \to I(n) \to \mathcal{O}_{\mathbb{P}^r \times S}(n) \to \mathcal{O}_X(n) \to 0$$

$n >> 0$ $\quad R^1 p_{2*} I(n)$ vanishes.

$$0 \to p_{2*} I(n) \to p_{2*} \mathcal{O}_{\mathbb{P}^r \times S}(n) \to p_{2*} \mathcal{O}_X(n) \to 0$$

clearly, $p_{2*}\mathcal{O}_{\mathbb{P}^r \times S}(n) = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \otimes \mathcal{O}_S$ is free ①

By flatness of $\mathcal{O}_X/S$, $p_{2*}(\mathcal{O}_X(n))$ is locally free

$\Rightarrow$ $p_{2*}\mathcal{I}(n)$ is locally free

$\Rightarrow$ $\mathcal{I}$ is flat $/S$. ②

Now $p_{2*}\mathcal{I}(d)$, d degree ...

Since for any hypersurface Y of degree d in $\mathbb{P}^r$,
$H^0(\mathbb{P}^r, \mathcal{I}_Y(d))$ is one dimensional, the theory of base change $\Rightarrow$ $p_{2*}\mathcal{I}(d)$ is a line bundle on S.

Thus, we can find a finite cover $\{U_i\}$ of S and a generator σ_i of $p_{2*}\mathcal{I}(d)$ on each U_i.

Each σ_i is a homogeneous polynomial of degree d

$$\sum_I \underbrace{b_{i,I}}_{\text{regular functions on } U_i} x^I$$

For any $s \in U_i$, $f^{-1}(s)$'s equation as a subscheme of $\mathbb{P}^r$ is precisely $\sum_I b_{i,I}(s)\, x^I = 0$

[Hint: $\mathcal{I}$ is the ideal sheaf of X]

We then define $\alpha_i : U_i \to \mathbb{P}^N$

$$s \mapsto [\dots : b_{i,I}(s) : \dots]$$

α_i : coordinate ring of $\mathbb{P}^N$ $\longrightarrow$ coordinate ring of U_i

$$a_I \longmapsto b_{i,I}$$

they glue : $b_{j,I} = u\, b_{i,I}$ for any multi index

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u unit on $U_i \cap U_j$.
to give a map: $\alpha: S \to \mathbb{P}^N$.