Lecture 16: Properties of flat families and Hilbert scheme

Wednesday, April 29, 2020 3:46 PM

Next week Topic for Presentation: Serre's GAGA
Reference: Algebraic and Analytic Geometry by Neeman
Subtopics: 1. Analytification of affine schemes
2. Analytification of schemes
3. Analytification of coherent sheaves
4. Main theorem: Serre's GAGA

Prop^n: let $f: X \to Y$ be a morphism of schemes, with Y integral and regular of dim 1. Then f is flat iff every associated point $x \in X$ maps to the generic pt. of Y. In particular if X is reduced, f is flat iff every irred. comp. of X dominates Y.

Proof: Suppose f is flat and let $x \in X$ such that $f(x) = y$ is a closed point of Y

$\Rightarrow \mathcal{O}_{Y,y}$ is a DVR.

let $t \in m_y \setminus m_y^2$ uniformizing element.

Then t is not a zero divisor in $\mathcal{O}_{Y,y}$. Since f is flat $f^\# t \in m_x$ is not a zero divisor, so x is not an associated pt. of X.

Converse, suppose that every ass. pt. maps to the generic pt. of Y.

Claim: f is flat.

$x \in X$, $y = f(x)$, $\mathcal{O}_{X,x}$ is flat / $\mathcal{O}_{Y,y}$

If y is the generic point, $\mathcal{O}_{Y,y}$ is field, ntp.

If y is a closed point, $\mathcal{O}_{Y,y}$ is a dvr, so we need to show $\mathcal{O}_{X,x}$ is torsion free

If it is not, $\Rightarrow$ $f^\# t$ must be a zero divisor in m_x

$\Rightarrow f^\# t \in \mathfrak{p} \in \mathrm{Ass}(\mathcal{O}_{X,x})$ $\quad \langle t \rangle = m_y$

Then $\mathfrak{p}$ determines a point $x' \in X$ which is associated point of X and $f(x') = y$ (not a generic pt.) $\Rightarrow\Leftarrow$ $\square$

X reduced $\mathrm{Ass}(X) = $ Generic pts. of X.

⚠ Result fails if Y dim > 1

Example, $Y = \mathbb{A}^2$, $X = $ blow up at a point in $\mathbb{A}^2$.

Then X, Y are nonsingular and X dominates Y

but $f: X \to Y$ is not flat!

— (dimension of fibers are not same.)

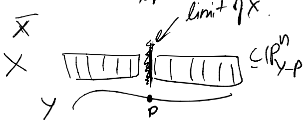
$$\begin{tikzcd} \bar{X} \arrow[r, phantom, "\subseteq"] & \mathbb{P}^n_Y \arrow[d] \\ X \arrow[u] \arrow[r, phantom, "\subseteq"] \arrow[dr, "\text{flat}"'] & \mathbb{P}^n_{Y-P} \arrow[u] \arrow[d] \\ & Y-P \end{tikzcd}$$

Prop^n: let Y be a regular, integral scheme of dim 1, let $P \in Y$ be a closed pt. and let $X \subseteq \mathbb{P}^n_{Y-P}$ be a closed subscheme which is flat over $Y-P$.

Then there exists a unique closed subscheme $\bar{X} \subseteq \mathbb{P}^n_Y$, flat over Y, whose restriction to $\mathbb{P}^n_{Y-P}$ is X.

Proof: $\bar{X} := $ closure of X in $\mathbb{P}^n_Y$

Associated points of $\bar{X}$ are just those of X, so the above Prop^n $\Rightarrow \bar{X}$ is flat Y.



Uniqueness: Any other ext^n of X to $\mathbb{P}^n_Y$, it would have some associated pt. mapping to P. $\square$.

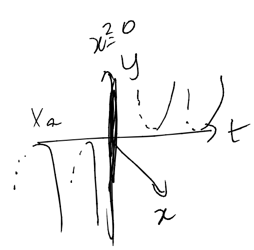
Remark: This will imply "Hilbert scheme" is proper.

⚠ In a flat family, irreducibility / reducedness is not preserved!!

Example: $k = \bar{k}$ let $X = \mathrm{Spec}\, k[x,y,t]/(ty - x^2)$
$Y = \mathrm{Spec}\, k[t]$

$f: X \to Y$ $\quad k[t] \hookrightarrow k[x,y,t]/(ty - x^2)$

- X, Y are integral schemes
- f is surjective $\Rightarrow$ dominant



Identify closed points of Y elements of k.

For $a \in k$, $a \neq 0$, $X_a :=$ plane curve $ay = x^2$ in $\mathbb{A}^2_k$ (irreducible, reduced).

But for $a = 0$, $X_0 := (x^2 = 0) \in \mathbb{A}^2_k$ (not reduced)

– $X = \mathrm{Spec}\, k[x,y,t]/(xy - t) \longrightarrow \mathrm{Spec}\, k[t] = Y$

$X_a = (xy = a)$ irreducible ——)(

$X_0 \quad (xy = 0)$ reducible. +

$X_a = (xy=a)^{\sim}$ irreducible —— $\mathcal{X}$
$X_0 \quad (xy=0)$ reducible. $+$

<u>Theorem</u>: Let T be an integral noetherian scheme. Let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme. For each point $t \in T$, consider the Hilbert polynomial $P_t \in \mathbb{Q}[z]$ of the fibre X_t considered as a closed subscheme of $\mathbb{P}^n_{k(t)}$. Then X is flat over T iff the Hilbert polynomial P_t is independent of t.

$\dim X = s \quad X \subseteq \mathbb{P}^r \quad \mathcal{F}$ coherent sheaf

Pf: <u>Hilbert polynomial</u>: This is a numerical polynomial characterizes

$$P_{X,\mathcal{F}}(m) = h^0(X, \mathcal{F}(m)) \quad m \gg 0.$$

- degree of X as the $s!$(leading term of the Hilbert polynomial)

$$\left(d \frac{m^s}{s!} \right)$$

- $P_t(m) = \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m)) \quad m \gg 0.$

First we generalize, replacing $\mathcal{O}_X$ by any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_T$, and using the Hilbert polynomial for $\mathcal{F}_t$.
Thus we may assume $X = \mathbb{P}^n_T$. $[0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0]$.

- We can assume $T = \operatorname{Spec} A$, A local Noetherian ring
[In fact, it is sufficient to compare any point to the generic pt.]

We will show TFAE:

(i) $\mathcal{F}$ is flat over $T = \operatorname{Spec} A$
(ii) $H^0(X, \mathcal{F}(m))$ is a free A-module of finite rank $\forall\, m \gg 0$
(iii) $P_t(\mathcal{F}_t)$ on $X_t = \mathbb{P}^n_{k(t)}$ is independent of t for any $t \in T$.

(i) $\Rightarrow$ (ii) Use Čech cohomology to compute $H^i(X, \mathcal{F}(m))$

$$H^i(X, \mathcal{F}(m)) = h^i(C^\cdot(\mathcal{U}, \mathcal{F}(m)))$$

Since $\mathcal{F}$ is flat $\Rightarrow C^i(\mathcal{U}, \mathcal{F}(m))$ is a flat A-module

But if $m \gg 0$ $H^i(X, \mathcal{F}(m)) = 0 \quad \forall i > 0$ (Serre's result)

Thus: $C^\cdot(\mathcal{U}, \mathcal{F}(m))$ is a resolution of $H^0(X, \mathcal{F}(m))$.

But if $m \gg 0$ $H^i(X, \mathcal{F}(m)) = 0$ $\forall i > 0$ (Serre's result)

Thus, $C^\cdot(\mathcal{U}, \mathcal{F}(m))$ is a resolution of $H^\circ(X, \mathcal{F}(m))$.

$$0 \to H^\circ(X, \mathcal{F}(m)) \to C^0(\mathcal{U}, \mathcal{F}(m)) \to C^1(\mathcal{U}, \mathcal{F}(m)) \to \cdots \to C^n(\mathcal{U}, \mathcal{F}(m)) \to 0$$

Split this complex into SES.

+ Use $0 \to M' \to \underbrace{M \to M''}_{\text{flat}} \to 0$

$\Rightarrow$ M' is flat

$\Rightarrow$ $H^\circ(X, \mathcal{F}(m))$ is flat A-module.

But it is also finitely generated (Serre's f.g of cohomology)

$\Rightarrow$ $H^\circ(X, \mathcal{F}(m))$ is free of finite rank.

(ii) $\Rightarrow$ (i) $S = A[x_0, \ldots, x_n]$ and $M = \bigoplus_{m \geq m_0} H^\circ(X, \mathcal{F}(m))$

$\mathbb{P}^n_A = \operatorname{Proj} S$ graded S-module

m_0 is large enough so that $H^\circ(X, \mathcal{F}(m))$ are all free for $m \geq m_0$.

Check: $\tilde{M} = \mathcal{F}$, $\tilde{M} = \widetilde{\Gamma_*(\mathcal{F})}$

Since M is free (and hence flat) $\Rightarrow$ $\mathcal{F}$ is flat/A. localizations)

(ii) $\Rightarrow$ (iii) We just need to show

$P_t(m) = \operatorname{rank}_A H^\circ(X, \mathcal{F}(m))$ for $m \gg 0$.

We will show for any $t \in T$, that

$$H^0(X_t, \mathcal{F}_t(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k(t). \quad \forall m \gg 0.$$

- $T' = \operatorname{Spec} A_{\mathfrak{p}} \xrightarrow[\text{flat}]{} T$ $\mathfrak{p} \mapsto t$. [Cohomology groups are preserved under flat base change]

and reduce to the case t is a closed point in T.

Denote the fibre X_t by X_0, $\mathcal{F}_t =: \mathcal{F}_0$, $k(t)$ is denoted by k.

Fix a presentation of k over A

$\to A \to k \to 0$ — (1)

Fix a presentation of k over A

$$A^{r} \longrightarrow A \longrightarrow k \to 0 \quad -(1)$$

We get an exact seq^n of sheaves

$$\mathcal{F}^{r} \to \mathcal{F} \to \mathcal{F}_0 \to 0$$

(Exercise) for $m \gg 0$ we get an exact sequence

$$H^0(X, \mathcal{F}(m)^{r}) \longrightarrow H^0(X, \mathcal{F}(m)) \to H^0(X_0, \mathcal{F}(m)) \to 0$$

Tensor (1) with $H^0(X, \mathcal{F}(m))$. Comparing

$$H^0(X_0, \mathcal{F}(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k \text{ for } m \gg 0.$$

(iii) $\Rightarrow$ (ii) Check $H^0(X, \mathcal{F}(m))$ is free.
by comparing its ranks at the generic point and the closed point of T. (Exercise: Reverse the above steps.) $\square$

Corollary: Let T be a connected noetherian scheme and let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme of $\mathbb{P}^n_T$ which is flat$/T$.
For any $t \in T$, let X_t be the fibre, considered as closed subscheme of $\mathbb{P}^n_{k(t)}$. Then the dim X_t, degree X_t, $P_a(X_t)$ (Arithmetic genus) are all independent of t.

Pf: Base change to irreducible components of T with their reduced induced structure, so are in the case T is integral. Then the result follows from the theorem above and

$$\dim X_t = \deg P_t$$

$r = \dim X$ $\quad \deg X_t = r!\,(\text{leading coeff of } P_t)$

$$P_a(X_t) = (-1)^r (P_t(0) - 1). \qquad \square$$

We define

$$\mathcal{M}_g : (Sch/\mathbb{C})^{opp} \longrightarrow Sets$$

$$S \longmapsto \left\{ \begin{array}{l} \text{proper flat family } X \to S \text{ where geom. fibers are connected smooth curves of genus } g \end{array} /iso \right\}$$

$\downarrow$

moduli of genus g curves

$\mathcal{M}_{g,n}$, $\bar{\mathcal{M}}_{g,n}$ ← stable curves

n-points.

<u>Hilbert Scheme</u> of $\mathbb{P}^r$

$$Hilb_r^{p(t)} : (Sch/k)^{opp} \longrightarrow Sets$$

$$S \longmapsto \left\{ \begin{array}{l} \text{flat families of closed subschemes } Z \subseteq \mathbb{P}^r_S \\ \text{with Hilbert polynomial of fibers as } p(t) \end{array} \right\} / \text{upto iso.}$$

r is fixed

$p(t)$ is fixed.

Z_s $\quad$ $Z \downarrow s \in S$ geom.

$Z_s \subseteq \mathbb{P}^r_{k(s)}$ | There are more geom. pt. $k(s)$

<u>Remark</u>: We can check flatness only at closed points!

$Z \subseteq \mathbb{P}^r$. closed immersions are preserved under base change.

$\downarrow$

$Spec(k) \to S$

<u>Thm</u>: Hilbert functor is representable by a projective scheme $Hilb_r^{p(t)}$, called the Hilbert scheme of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$.

$[Y] := Y \overset{closed}{\subseteq} \mathbb{P}^r$ with hilbert polynomial $p(t)$.

inside $Hilb^{p(t)}$

$[Y] := Y \subseteq \mathbb{P}^r$ with Hilbert polynomial ...

inside $\mathrm{Hilb}_r^{P(t)}$.

Example: $X \subseteq \mathbb{P}^r$ hypersurface of fixed degree d.

Then

— Hilbert polynomial $p_X(n) = \binom{n+r}{r} - \binom{n-d+r}{r}$

[Ex: Hint:

$0 \to \mathcal{O}_{\mathbb{P}^r}(n-d) \to \mathcal{O}_{\mathbb{P}^r}(n) \to \mathcal{O}_X(n) \to 0$]

— $P_X(n)$ does not depend on a particular X but only on degree of X and leading term $\frac{d t^{r-1}}{(r-1)!}$

— Ex: Show that this Hilbert polynomial characterizes hypersurfaces $Y \subseteq \mathbb{P}^r$ among all subschemes of $\mathbb{P}^r$ having Hilbert polynomial $p(n)$.

Let $X \subseteq \mathbb{P}^r$ be a subscheme s.t $P_X(n) = p(n)$

Then X is a hypersurface.