§1 : Functor of points :

Not^n: $\mathcal{C}$ category $\mathcal{C}^{opp}$ opposite category
same objects, arrows reversed
$\mathrm{Hom}_{\mathcal{C}^{opp}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(y,x)$
$\mathrm{Mor}_{\mathcal{C}^{opp}}$ $\mathrm{Mor}_{\mathcal{C}}$

Def^n: Let $\mathcal{C}, S$ be two categories.
A contravariant functor F from $\mathcal{C}$ to S is a functor
$$F : \mathcal{C}^{opp} \to S$$
$$F : Ob(\mathcal{C}) \longrightarrow Ob(S)$$
and for every $\psi : x \to y \longmapsto F(\psi) : F(y) \to F(x)$
(composition is reversed as well)

Defn: Let $\mathcal{C}$ be a category.
(1) A presheaf of "set" on $\mathcal{C}$ is a contravariant functor F from $\mathcal{C}$ to sets (cat. of sets).
(2) The category of presheaves as $PSh(\mathcal{C})$.

Example: Functor of points For $U \in Ob(\mathcal{C})$
$$h_U : \mathcal{C} \longrightarrow \text{sets}$$
contravariant $X \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, U)$

also denoted $h_U : \mathcal{C}^{opp} \to \text{sets}$

S scheme (Sch) $\supseteq$
$\mathcal{C} = (Sch/S)$ $U \in (Sch/S)$
$h_U : (Sch/S)^{opp} \to \text{sets}$
Functor of points of scheme U.
Ex: Show that all k-points of a scheme U are $h_U(k)$.

Also called representable presheaf associated to U.

Note that $\phi : U \to V$ in $\mathcal{C}$ we have $h(\phi) : h_U \to h_V$
morphism (defined by $h_U(T) \to h_V(T)$
$\mathrm{Hom}(T,U)$ $\mathrm{Hom}(T,V)$
$T \to U \mapsto T \xrightarrow{\phi} V$

This gives us a functor to the category of functors
$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{opp}, \text{sets}) = PSh(\mathcal{C})$$
"big category" ← set theoretic remark...

(Yoneda lemma) : let $U, V \in Ob(\mathcal{C})$. Given any morphism of functors $s : h_U \to h_V$ there is a unique morphism $\phi : U \to V$ such that $h(\phi) = s$.
In otherwords, the functor h is fully-faithful.
More generally, given any contravariant functor F and any object U of $\mathcal{C}$ we have a natural bijection
$$\mathrm{Hom}_{PSh(\mathcal{C})}(h_U, F) \longrightarrow F(U)$$
$$s \longmapsto s_U(id_U).$$

Pf: $s_U : h_U(U) \longrightarrow h_V(U)$
$\mathrm{Hom}(U,U) \longrightarrow \mathrm{Hom}(U, V)$
$id \longmapsto s_U(id_U)$

Take $\phi = s_U(id_U)$.

For the second statement, given $\xi \in F(U)$, define s by
$$s_V : h_U(V) \longrightarrow F(V)$$
$$f : V \to U \longmapsto F(f)(\xi).$$
$\square$

Defn. A contravariant functor

$f: V \to U \mapsto F(f)(\xi)$. □

Def^n: A contravariant functor
$F: \mathcal{C} \to Sets$
is said to be representable if it is isomorphic to the functor of points h_U for some U of $\mathcal{C}$.

$$F \cong h_U.$$

Remarks: $\mathcal{C}$: $F: \mathcal{C}^{opp} \to Sets$ is a rep^n functor.
Choose an object U of $\mathcal{C}$ and an iso $s: h_U \to F$.
Then Yoneda lemma guarantees that the pair (U, s) is unique upto a unique morphism.
U is called object representing F.
By Yoneda lemma the transformation s corresponds to a unique element $\xi \in F(U)$

universal object.

It has the property that for $V \in Ob(\mathcal{C})$, the maps
$$Mor_{\mathcal{C}}(V, U) \longrightarrow F(V)$$
$$(f: V \to U) \longmapsto F(f)(\xi)$$
is a bijection.

Thus ξ is universal in the sense that every element of $F(V)$ is equal to the image of ξ via $F(f)$. for a unique morphism $f: V \to U$ in $\mathcal{C}$.

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let $\mathcal{C} = (Sch) / (Sch/S)$.

Def^n: A presheaf $F: (Sch) \to Sets$ is said to be a sheaf in the Zariski topology if for every scheme T and every open covering $T = U_i, i \in I$ and for any collection of elements $\xi_i \in F(U_i)$ s.t $\xi_i|_{\dots} = \xi_j|_{U_{ij}}$ there any open covering $T = \bigcup U_i, i \in I$

of elements $\xi_i \in F(U_i)$ s.t $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exist a unique element $\xi \in F(T)$ s.t $\xi_i = \xi|_{U_i}$ in $F(U_i)$.

Exercise: Show that h_U for any scheme U is a sheaf in the Zariski topology. In particular, deduce that for a presheaf to be representable it must be a sheaf in the Zariski topology.

Moduli problem:

Aim: We want to classify a collection of objects ("geom/alg") (in our case a collection of curve) over an algebraically closed field upto certain equivalence (isomorphism)

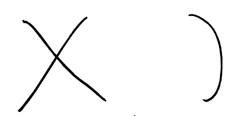
$\mathcal{M}_g :=$ set of curves of genus $g / k = \bar{k}$ upto isomorphism.

structure of a scheme over this set.

— Notion of family.

Geometry of $\mathcal{M}_g$ to relate isomorphism classes of curves in some way.

"limits"


"deformation"

Remark: Smooth projective curves $/\mathbb{C}$

Thm (Torelli): X, Y are curves as above. Then $X \cong Y$ iff $\mathrm{Jac}(X) \cong \mathrm{Jac}(Y)$ (isomorphism in category of schemes)

structure of an abelian variety.

$\mathrm{Jac}(X) = \mathrm{Pic}^0(X) :=$ set of degree zero line bundles on X.

$\mathcal{M}_g :=$ Notion of a family

Family := $\pi : X \to B$ morphism of schemes s.t. $\pi^{-1}(b)$ is a scheme. $b \in B$ point

This is too general (useless), fibers have nothing in common

$\pi : X \to B$ be a family $b \in B$ be a closed point.

New family $\pi' : (X \setminus \pi^{-1}(b)) \sqcup Y \longrightarrow B$

$$Y \longmapsto b \qquad Y \text{ any scheme}$$

$$X_{\setminus \pi^{-1}(b)} \longmapsto B \setminus b$$

We have to put some conditions on the type of morphisms to be allowed.

Ans: The best current candidate is flatness.

- local triviality of family $X \times B \to B$ is bad in the locally coarse Zariski topology
- local completions $\xleftrightarrow[\text{GAGA}]{}$ local analytic rings.

only smooth family...

To pose a moduli questions:

1. A class of geometric objects (scheme, line bundles, vector bundles.) Perfect complexes
2. A notion of family.
3. A notion of equivalence

Defn: A moduli problem for a class P of objects in a category $\mathcal{C}$ is a contravariant functor

$$F_P : \mathcal{C} \longrightarrow \text{sets}$$

that associates to each $B \in \mathcal{C}$ the set of isomorphism classes of families of P-objects.

To each morphism $f : B' \to B$ it associates the set sending a family $X \to B$ to the pullback family $f^*(X) \to B'$

Def: A object $M_P \in \mathcal{C}$ that represents the functor F_P is called a fine moduli space.

Def: A object $M_p \in C$ that represents the functor,
is called a <u>fine moduli space</u>.

Rmk.. Yoneda $\Rightarrow$ Fine moduli space is unique upto unique isomorphism.

Back to our notion of family in case of schemes.

- Atleast dimension of fibers should be same
- Certain (numerical) invariants be constant in the fibers of the family (genus, Hilbert polynomial)

Prop^n: Let $f: X \to Y$ be a flat morphism of schemes of finite type over a field k. For any point $x \in X$, let $y = f(x)$. Then $\dim_x(X_y) = \dim_x X - \dim_y Y$.

$\dim_x X := \dim \mathcal{O}_{X,x}$

Proof: Reduction to the case y is a closed point of Y.

Take $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}$.

Base change

$$\begin{tikzcd} x' \in X' \arrow[r] \arrow[d, "flat"'] & X \ni x \arrow[d, "flat"] \\ y \in Y' = \operatorname{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y \ni y \end{tikzcd}$$

closed point

Check: $\dim_x X' = \dim_x X$

$\dim_x (X'_y) = \dim_x (X_y)$

$\dim_y Y = \dim_y Y'$.

Now we use induction on $\dim Y$.

If $\dim Y = 0$. (Ex.) X_y is defined by a nilpotent ideal in X.

we have $\dim_x(X_y) = \dim_x X$ and $\dim_y Y = 0$.

If $\dim Y > 0$, base change to Y_{red} (dim don't change)

assume Y is reduced.

Then find an element $t \in m_y \in \mathcal{O}_{Y,y}$ s.t t is a non-zerodivisor

Let $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}/(t)$ and make the base change to $Y' \to Y$.

(Ex) $\dim Y' = \dim Y - 1$,

Since f is flat, $f^\# t \in m_x$ is not a zero divisor

so for the same reason $\dim_x X' = \dim_x X - 1$.

But X_y does not change under base change to Y' $\square$

Corollary: let $f: X \to Y$ be a flat morphism of schemes of f.t over a field k and assume that Y is irreducible.

TFAE:

(i) every irreducible component of X has dim equal to $\dim Y + n$.

(ii) for any point $y \in Y$ (closed or not), every irreducible component of the fibre X_y has dimension n.

Pf: (i) ⇒ (ii) Given $y \in Y$, $Z \subseteq X_y$ irr comp. and let $x \in Z$ be a closed pt. s.t. $x \notin Z'$ for any other irr. component of X_y.

Applying the previous result

$$\dim_x Z = \dim_x X - \dim_y Y.$$

Now $\dim_x Z = \dim Z$ (x is closed)

Since Y is irreducible. X is equi-dimensional + f.t.

$$\operatorname{cod}(Z, X) = \dim X - \dim Z$$

$$\dim_x X = \dim X - \dim \overline{\{x\}} \quad [Ex.]$$

$$\dim_y Y = \dim Y - \dim \overline{\{y\}}$$

x was closed in X_y, $k(x)$ is a finite algebraic ext^n of $k(y)$ and so

$$\dim \overline{\{x\}} = \dim \overline{\{y\}}$$

Combining this with (i) we get $\dim Z = n$.

(ii) ⇒ (i) Exercise.

Prop^n: let $f: X \to Y$ be a morphism of schemes, with Y integral and regular of dim 1. Then f is flat iff every associated point $x \in X$, maps to the generic point of Y. In particular, if X is reduced, f is flat iff every irreducible component of X dominates Y.