Lecture 14: Curves of Higher Genus

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(Genus 2 and 3)

Def^n: A projective regular integral genus g curve C is hyperelliptic if it admits a double cover of $\mathbb{P}^1_k$ (i.e. degree 2 finite morphism to $\mathbb{P}^1$) $\pi: C \to \mathbb{P}^1_k$

Hurwitz's formula $\Rightarrow$ π is ramified at $2g+2$ points

Qus: Do such curves exist?

Prop^n: Assume char $k \neq 2$, $k = \bar{k}$. Given r distinct points $p_1, \dots, p_r \in \mathbb{P}^1$, there is precisely one double cover branched at these points if r is even and non if r is odd.

Proof: Pick points 0 and ∞ in $\mathbb{P}^1$ distinct from the r branch points

$\Rightarrow$ All r branch points $\in \mathbb{P}^1 \setminus \infty = \mathbb{A}^1 = \operatorname{Spec} k[x]$.

Suppose $C' \longrightarrow \mathbb{A}^1$ double cover of $\mathbb{A}^1 = \operatorname{Spec} k[x]$

This induces a quadratic field ext^n K over $k(x)$

Char $k \neq 2$ $\Rightarrow$ $k(x) \hookrightarrow K$ is Galois. $[K : k(x)] = 2$

$\sigma: K \longrightarrow K$ non-trivial Galois involution.

$0 \neq y \in K$ st. $\sigma(y) = -y$, so $1, y$ generate $K / k(x)$

Now $\sigma(y^2) = y^2$ $\quad y^2 \in k(x)$

Replace y by αy, $\quad y^2 = \underbrace{x^N + a_{N-1} x^{N-1} + \dots + a_0}_{\text{no repeated roots}}$ $\quad C_0'$ curve in $k[x,y]$

- check: use Jacobian criterion C_0' is regular curve.

$\Rightarrow$ C_0' is normal and $K(C_0') = K(C')$

Thus C_0' and C' are both normalizations of $\mathbb{A}^1 \subset$ finite ext^n field gene. by y

and $C_0' \cong C'$.

- The branch points correspond to values of x for which there is exactly one value of y, i.e. roots of $f(x)$.

In particular $N = r$, and $f(x) = (x - p_1) \dots (x - p_r)$.

where $p_i \in k$.

- Affine open set $\mathbb{P}^1 \setminus \{0\} = \operatorname{Spec} k[u]$ where $u = 1/x$.

double cover $C'' = \operatorname{Spec} k[z, u] / (z^2 - (u - 1/p_1) \dots (u - 1/p_r))$

$= \operatorname{Spec} k[z,u] / ((\prod p_i) z^2 - (1-u)^r f(1/u))$

$\downarrow$

$\operatorname{Spec} k[u] = \mathbb{A}^1.$

So if there is a double cover over all of $\mathbb{P}^1$, it must be obtained by gluing C'' to C' over the gluing of $\operatorname{Spec} k[x]$ to $\operatorname{Spec} k[u]$.

In $K(C)$, we must have

$$z^2 = u^r f(1/u) = f(x)/x^r = y^2/x^r.$$

If r is even: Consider $K(C)$ generated by y and x
$z := \pm y/x^{r/2}$ (take +ve square root).

If r is odd: - Note that x does not have a square root in the field $k(x)[y]/(y^2 - f(x))$
$\prod (x - p_i)$ $p_i \neq 0$

- This proves that in case r is odd no double covers ramified at r points exist. $\square$.

$\Rightarrow$ There are curves of every genus $g \geq 0$ over an algebraically closed field of char $k \neq 2$:

To get a genus g curve, consider the double cover branched over $2g+2$ distinct points.

Lemma A: A curve as above of genus ≥ 1 is hyperelliptic iff there exists a Cartier divisor D on X s.t $\ell(D) = 2 = \deg D$.

Proof: ($\Leftarrow$) Suppose $D \in \operatorname{Div}(X)$ exists
As $\ell(D) \neq 0$, we can reduce to case $D \geq 0$ by linear equivalence

let $x \in \operatorname{Supp} D$ - If $\deg(D-x) \leq 0$,
$\ell(D-x) \leq 1$ (?)

$\deg(D-x) = 0$ / $\deg(D-x) < 0$ ← no global section
$\ell(D-x) = 0$.

$\ell(D-x) = 0$ $\quad$ $\ell(D-x) \neq 0$.

$\Rightarrow \mathcal{O}_X(D-x) \cong \mathcal{O}_X$

... integral

$\Rightarrow \mathcal{O}_X(D-x) \cong \mathcal{O}_X$

Since we are smooth & integral $H^0(X, \mathcal{O}_X) = k$.

$\ell(D-x) = 1$

- If $\deg(D-x) = 1$ we have $\ell(D-x) \leq 1$

otherwise $\Rightarrow X \cong \mathbb{P}^1 \Rightarrow \Leftarrow$

It follows from Lemma 7.4.2 (Liu) that $\mathcal{O}_X(D)$ is generated by its global sections.

$\Rightarrow \exists\, \pi : X \longrightarrow \mathbb{P}^1_k$ since $\ell(D) = 2$

check: $\deg \pi$ is $2 = \deg D$.

- π is separable: π deg 2 if it is not sep. it is purely insep. $\Rightarrow \Leftarrow$ $g(X) \neq 0$.

Conversely, suppose that we have a morphism $\pi : X \to \mathbb{P}^1_k$ of degree 2. Fix a rational point y_0 of $\mathbb{P}^1_k$, considered as a Cartier divisor and let $D = \pi^* y_0 \in \mathrm{Div}(X)$.

$\deg D = 2$.

$L(D) \supset H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(y_0))$, we have $\ell(D) \geq 2$.

<u>Claim</u>: $\ell(D) \leq 2$, let $x \in \mathrm{Supp}\, D$

then $\ell(D) \leq \deg(D-x) + \ell(x) = 2$ $\square$

<u>Prop^n</u>: Let X be a smooth, integral, projective curve. We suppose that X is elliptic or of genus 2. Then X is hyperelliptic curve.

Proof: $P_0 \in X$ elliptic, Use $D = 2P_0$

<u>genus 2</u>: let K be a canonical divisor of X. Then $\deg K_X = 2g-2 = 2$ and $\ell(K_X) = p_a = 2$ and use Lemma A. $\square$

<u>Exercise</u>: In Lemma A above, $D \in \mathrm{Div}(X)$ inducing a hyperelliptic cover, show that $\mathcal{O}_X(D)^{\otimes(g-1)} \cong \omega_X \leftarrow$ canonical sheaf of X.

<u>Qus</u>: When is your line bundle ω_X?

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Suppose $\mathcal{L}$ is a degree $2g-2$ line bundle. Then

Claim: $\underset{\overset{\shortparallel}{\ell(D)}}{\ell(\mathcal{L})} = g$ or $g-1$. $\quad \mathcal{L} = \mathcal{O}_X(D)$

R-R: $\ell(D) - \ell(K-D) = \underset{\underset{2g-2}{\shortparallel}}{\deg D} + 1 - g$.

as $\deg(K-D) = 0 \Rightarrow \ell(K-D) = 1$ or 0

$\Rightarrow \ell(D) = g-1+\ell(K-D)$

$= \begin{cases} g-1 & \ell(K-D)=0 \\ g & \ell(K-D)=1. \end{cases}$

$\ell(K-D) \neq 0 \Rightarrow \mathcal{O}_X(K-D) \cong \mathcal{O}_X$

$\Rightarrow H^0(X, \mathcal{O}_X(K-D)) = 1$

In case we have g dim - global sections:

$\ell(K-D) = 1$. $\quad \mathcal{O}_X(K-D) \cong \mathcal{O}_X$

$\omega_X \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$

$\Rightarrow \mathcal{L} \cong \omega_X$ $\square$.

Hint: compose the hyperelliptic map

$$\underbrace{X \xrightarrow[|\mathcal{L}|]{} \mathbb{P}^1 \xrightarrow[|\mathcal{O}_{\mathbb{P}^1}(g-1)|]{} \mathbb{P}^{g-1}}_{\mathcal{L}^{\otimes(g-1)}}$$

$H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) \hookrightarrow H^0(C, \mathcal{L}^{\otimes(g-1)})$

injection. $\square$.

Propⁿ: (A genus $g \geq 2$ curve can be hyperelliptic in only one way!)
(Exercise)

Any curve X of genus at least 2 admits at most one double cover of $\mathbb{P}^1$. More precisely, if $\mathcal{L}$ and $\mathcal{M}$ are two degree 2 line bundles yielding maps $C \to \mathbb{P}^1$ then $\mathcal{L} \cong \mathcal{M}$.

- Every genus ≥ 2 curve has finite automorphism group.

"Informal": moduli space dim $= 2g+2 - \dim \underline{\mathrm{Aut}(\mathbb{P}^1)}$
genus $2 =$ "3"

1 § Curves of genus 3:

Propⁿ: Let X be a smooth integral, projective curve over $k = \bar{k}$ of genus ≥ 1. Then $\Omega^1_{X/k}$ is generated by global sections.

Proof: K canonical divisor
$l(K) = g > 0$, assume wlog $K \geq 0$
Let $x \in X(k)$. Then $l(x) = 1$ (as $X \not\cong \mathbb{P}^1$)
Then $l(K-x) = \deg(K-x) + l(x) + 1 - g$
$= 2g - 2 - g + 1 = g - 1 < l(K)$.

So generated by global section.

Defⁿ: X as above. The morphism $X \to \mathbb{P}^{g-1}$ defined by $\Omega^1_{X/k}$ (and the choice of a basis of $H^0(X, \Omega^1_{X/k})$) is called the canonical map.

curve of genus $g \geq 2$

Canonical map.

Prop^n: Let X be smooth integral proj. curve of genus $g \geq 2$
$k = \bar{k}$. Then the canonical map $X \to \mathbb{P}^{g-1}$ is a closed immersion iff X is not hyperelliptic.

Proof: K canonical divisor
Suppose X is not hyperelliptic. Let $E \in \mathrm{Div}_+(X)$ of deg 2

Lemma A $\Rightarrow l(E) = 1$.

R-R $\Rightarrow l(K-E) = l(E) + \deg(K-E) + 1-g$
$= g-2$.

$\Rightarrow$ we have a closed embedding. [Lemma 7.4.3]

$\Rightarrow$ K is very ample.

Converse: if X is hyperelliptic, $\omega_C \cong \mathcal{L}^{\otimes(g-1)}$

$$\omega_C : X \xrightarrow[\text{double cover}]{} \mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$$ conic. $\square$

In case: $g=3$, if C is not hyperelliptic
$\deg \omega_C = 2g-2 = 4$, $l(\omega_C) = 3$

We can describe C as a degree 4 curve in $\mathbb{P}^2$.

Converse any quartic plane curve is canonically embedded. Reason: the curve has genus 3

$g = \frac{(4-1)(4-2)}{2} = 3$

embedding: given by line bundle of degree 4
and 3 independent global sections
such an $\mathcal{L} \cong \omega_C$. $\square$

In conclusion: {non-hyperelliptic genus 3 curves} $\xleftrightarrow{1-1}$ {plane quartic curves up to projective linear transformations}

$\boxed{2}$ "hyperelliptic $\to$ dim 5" "plane quartics" $\leftrightarrow$ 6 dim family"
"single family of 6 dim".

Infact, hyperelliptic curves are naturally limit of non-hyperelliptic curves

<u>sketch</u>: $\pi : C \to \mathbb{P}^1$ branched over $2g+2 = 8$ points

$g = 3$

Choose an iso of $\mathbb{P}^1$ with a conic in $\mathbb{P}^2$

There is a regular quartic meeting the conic in precisely 8 points (Use Bertini's Thm).

Then if f is the eq^n of conic, $g :=$ quartic

$f^2 + t^2 g$ "family of quartics that are smooth for most t"

The case $t = 0$, is a double conic.

Then normalize the total space of the family.

then the central fiber (above $t=0$) turns into hyperelliptic curve.

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