Lecture 12: Hurwitz's Theorem

Wednesday, April 1, 2020 3:54 PM

We will take two week Easter break as per schedule. Next class on 20th April.
Meanwhile, here is the reading assignment + a few exercises:
Reading assignment: Section IV.3 Hartshorne, embedding of a smooth curve in P^3.
Exercises: Hartshorne IV.2.1, 2.2, 2.3 (page 304).

Recall: $D \in \mathrm{Div}(\mathbb{P}^n_k) \xrightarrow{\rho} \mathrm{CaCl}(\mathbb{P}^n_k) \cong \mathrm{Cl}(\mathbb{P}^n)$

$\rho \downarrow$, $\delta[D]$, $\uparrow$ $\mathbb{Z}$ — degree of a divisor.

$\mathrm{Cl}(\mathbb{P}^n) \underset{\text{IS}}{\cong} \mathbb{Z}\cdot\xi$ — hyperplane class

Lemma: Let X be a closed subvariety of $\dim \geq 1$ of $P := \mathbb{P}^d_k$. We suppose X is integral and a complete intersection in $\mathbb{P}^n$. Let $D \in \mathrm{Div}_+(X)$ s.t. $\mathcal{O}_X(D) \simeq i^*\mathcal{O}_{\mathbb{P}^d}(m)$, where $i : X \hookrightarrow \mathbb{P}^d$ is the canonical injection and where $m \in \mathbb{Z}$. Then $m \geq 0$ and there exists an effective divisor H on $\mathbb{P}^d$ such that $\deg H = m$ and $H|_X = D$.
In particular, $(\mathrm{Supp}\, H) \cap X = \mathrm{Supp}\, D$.

Proof. $H' \in \mathrm{Div}(\mathbb{P}^d)$ not containing the generic pt. of X and of degree m.

$\Rightarrow H'|_X$ is an Cartier-divisor on X.

[Note: $i : X \hookrightarrow Y$ Noeth.

Ex: $\{E \in \mathrm{Div}(Y) \mid \mathrm{Supp}(E) \cap \mathrm{Ass}(\mathcal{O}_X) = \emptyset\} \to \mathrm{Div}(X)$

$E \longmapsto E|_X$

compatible with the hom $\mathcal{O}_Y \to i_*\mathcal{O}_X$

$\mathrm{Supp}(E|_X) = \mathrm{Supp}(E) \cap X$, $\mathcal{O}_Y(E)|_X \simeq \mathcal{O}_X(E|_X)$.

If $E \geq 0$, $E|_X \geq 0$, $\mathrm{div}(t)|_E$ restrict well.]

$\exists$ a $t \in k(X)^*$ such that $\underline{H'|_X + \mathrm{div}(t)} = D$

Hence $t \in L(H'|_X)$ as D is effective.

Consider the homomorphism

$L(H') \longrightarrow L(H'|_X)$ deduced from

$\phi : \mathcal{O}_{\mathbb{P}^d}(H') \longrightarrow i_* i^* \mathcal{O}_{\mathbb{P}^d}(H') \simeq i_*\mathcal{O}_X(H'|_X)$ by taking global sections.

upto iso identical to

$\mathcal{O}_{\mathbb{P}^d}(m) \longrightarrow i_* i^* \mathcal{O}_{\mathbb{P}^d}(m)$

$H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(m)) \longrightarrow H^0(\mathbb{P}, i_* i^* \mathcal{O}_{\mathbb{P}}(m))$

is surjective (Exercise).

$\Rightarrow L(H') \longrightarrow L(H'|_X)$ is surjective.

$s \longmapsto t$

$H := H' + \mathrm{div}(s) \geq 0$, $\deg H = \deg H'$, so $n \geq 0$

$H|_X = H'|_X + \mathrm{div}(s)|_X = D_B$

Corollary: Let X be a smooth, connected projective curve $/k$ of genus 1. Suppose $\exists\, o \in X(k)$. Then X is an elliptic curve and $\mathcal{O}_X(3o)$ is very ample.

Pf: $X(k) \neq \Phi \Rightarrow X$ is geom. connected. (Exercise)

Since $3o$ is effective $\Rightarrow \mathcal{O}_X(3o)$ is very ample.
$\ 2g+1$

As $l(D) = 3$, $\mathcal{O}_X(3o)$ induces a closed embedding into $\mathbb{P}^2_k$. The genus formula $\frac{(n-1)(n-2)}{2} = 1 \Rightarrow n = 3$

$\Rightarrow$ Equation for X is a cubic.

$X = V_+(F(u,v,w))$ F homogeneous of degree 3

H be a line in $\mathbb{P}^2_k$ such $H \cap X = \{o\}$.
(use the lemma above with $n=1$, $D = 3o$, $\mathbb{P}^2_k$)

By an automorphism of $\mathbb{P}^2_k$, suppose $o = (0:1:0)$
$H = \{w = 0\}$

Then multiplying u, v by suitable elements of k^*

$F(u,v,w) = v^2w + (a_1u + a_3w)vw + (u^3 + a_2u^2w + a_4uw^2 + a_6w^3)$ □

Hurwitz's Theorem / Formula [Curves: nonsingular projective irreducible curves $/k = \bar{k}$]

Recall degree of a finite morphism of curves $f: X \to Y$ as the degree of $[K(X) : K(Y)]$

For any point $P \in X$, we define ramification index e_P as:

$Q = f(P)$, $t \in \mathcal{O}_Q$ local parameter (DVRs)
$\pi \in \mathcal{O}_P$ local parameter.

Consider t as an element of $\mathcal{O}_P$ via natural map $f^\#: \mathcal{O}_Q \to \mathcal{O}_P$ and

$t = u\pi^e$

define $e_P := \underbrace{v_P}_{\text{associated valuation}}(t) = e$

If $e_P > 1$ we say f is ramified at P

$e_P := v_P(t)$ ... associated valuation

If $e_P > 1$ we say f is ramified at P

& branch point of f.

~~or [k(P) is insep/k(Q)]~~

If $e_P = 1$, f is unramified $\iff$ f étale

If char $k = 0$ or if char $k = p$, and $p \nmid e_P$, then ramification is tame

If $p \mid e_P$, ramification is wild.

Recall: $f^* : \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(X)$ $f^*(Q) = \sum_{P \mapsto Q} e_P P$.

& $f^*(\mathcal{L}(D)) \simeq \mathcal{L}(f^*(D))$.

Defn: $f: X \to Y$ separable if $k(X)/k(Y)$ is a sep. field ext?

Propn: Let $f: X \to Y$ be a finite sep. morphism of curves. Then there is an exact sequence of sheaves on X,

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

(II 8.1) Hart.

Pf:

Claim: $f^*\Omega_Y \longrightarrow \Omega_X$ is injective.

Since both are invertible sheaves on X, (integral) it is sufficient to show that the map is non-zero at the generic point.

But since $k(X)/k(Y)$ is sep., the sheaf $\Omega_{X/Y}$ is zero at the generic point.

Hence $f^*\Omega_Y \to \Omega_X$ is surjective at generic pt. □

- For any point $P \in X$, let $Q = f(P)$, let t be a local parameter at Q, u local parameter at P.

Then dt is a generator of the free $\mathcal{O}_Q$-module $\Omega_{Y,Q}$

du ——— $\mathcal{O}_P$ - $\Omega_{X,P}$

In particular, there is a unique element $g \in \mathcal{O}_P$ s.t $f^*dt = g \cdot du$

We denote this g by dt/du.

Propn. Let $f: X \to Y$ be a finite, sep morphism of curve. Then

Prop^n: Let $f: X \to Y$ be a finite, sep morphism of curves. Then
a) $\Omega_{X/Y}$ is a torsion sheaf on X with support equal to the set of ramification points of f.
In particular f is ramified at only finitely many points.

Pf: $(\Omega_{X/Y})_P = 0$ iff $f^* dt$ is a generator for $\Omega_{X,P}$
$\iff$ t is a local parameter for $\mathcal{O}_P$
$\iff$ f is unramified at P. $\square$

b) for each $P \in X$ the stalk $(\Omega_{X/Y})_P$ is a principal $\mathcal{O}_P$-module of finite length equal to $v_P(dt/du)$.

c) if f is tamely ramified at P, then length $(\Omega_{X/Y})_P = e_P - 1$
If f is wildly ramified at P, then length $> e_P - 1$.

Proof: b) $(\Omega_{X/Y})_P \cong \Omega_{X,P} / f^* \Omega_{Y,P} \cong \mathcal{O}_P / (dt/du)$

(c) f has ramification index e.

$$t = a u^e \quad \text{for } a \in \mathcal{O}_P^\times$$

$$dt = a e u^{e-1} du + u^e da$$

If ramification is tame, then e is a non-zero element of k, so we have

$$v_P(dt/du) = e - 1$$

otherwise $v_P(dt/du) \geq e$ $\square$

Def^n: Let $f: X \to Y$ be a finite, sep. morphism of curves. Then we define the <u>ramification divisor</u> of f to be

$$R = \sum_{P \in X} \text{length}(\Omega_{X/Y})_P \cdot P$$

Prop^n: Let $f: X \to Y$ —— K_X, K_Y canonical divisors of X, Y.
Then $K_X \sim f^* K_Y + R$.

Then $K_X \sim f^* K_Y + R$.

Pf: $R \hookrightarrow X$ (closed subscheme)

Then note that $\mathcal{O}_R \simeq \Omega_{X/Y}$.

$\boxed{\Omega_{X/Y} \otimes \Omega^{-1}_{X/k} \text{ is } \Omega_{X/Y} \otimes \Omega^{-1}_{X/k}}$

$$0 \to f^* \Omega_{Y/k} \to \Omega_{X/k} \to \underbrace{\Omega_{X/Y}}_{\text{torsion sheaf}} \to 0 \quad \otimes \Omega^{-1}_{X/k}$$

$$0 \to \underbrace{f^* \Omega_{Y/k} \otimes \Omega^{-1}_{X/k}}_{\text{ideal sheaf of } R} \to \mathcal{O}_X \to \Omega_{X/Y} \otimes \Omega^{-1}_{X/k} \to 0$$

$f^* \Omega_{Y/k} \otimes \Omega^{-1}_{X/k} = \mathcal{O}_X(-R)$, $\quad \Omega_{X/Y} \otimes \Omega^{-1}_{X/k} \simeq \mathcal{O}_R$.

Take the associated divisor.

$K_X \sim f^* K_Y + R$. □

Corollary (Hurwitz Formula):
Let $f: X \to Y$ be a finite sep. morphism of curves

Let $n = \deg f$. Then

$$2g(X) - 2 = n(2g(Y) - 2) + \deg R.$$

Furthermore if f has only tame ramifications, then

$$\deg R = \sum_{P \in X} (e_P - 1).$$

Pf: Put in all the comp. from above. □

Application: ($\mathbb{P}^1$ is simply connected) / Alg. fundamental group.

An étale covering of a scheme Y is a scheme X, together with a finite étale morphism $f: X \to Y$.

It is called trivial if X is isomorphic to a finite disjoint union of copies of Y.

... connected if it has no non-trivial

union of copies of Y.

Y is called <u>simply connected</u> if it has no non-trivial étale coverings.

<u>Claim</u>: $\mathbb{P}^1$ is simply connected.

Indeed, let $f: X \to \mathbb{P}^1$ be an étale covering. assume X is connected

f étale $\Rightarrow$ X is smooth$/k$

f finite $\Rightarrow$ X proper $/k$

$\Rightarrow$ X is an irreducible non-singular projective curve $/k$.

f is étale $\Rightarrow f$ is sep.

Use Hurwitz, since f is unramified, $R=0$

$$2g(X)-2 = n(-2)$$

smooth $g(X) \geq 0$ $\Rightarrow$ $g(X)=0$ & $n=1$.

Thus $X = \mathbb{P}^1$.

• If $f: X \to Y$ is any finite morphism of curves, then $g(X) \geq g(Y)$.

$k(Y) \subset k(X)$ factorize this ext^n into purely insep + sep. parts. ↑ if char $= 0$

We are reduced to the case f is sep.

If $g(Y)=0$, ntp.

{such morphism do not change the genus of the curve} char p: [Frobenius morphism]

Assume $g(Y) \geq 1$.

Hurwitz formula:

$$g(X) = g(Y) + (n-1)(g(Y)-1) + \tfrac{1}{2}\deg R$$

$n-1 \geq 0$, $g(Y)-1 \geq 0$, $\deg R \geq 0$. $\square$

equality occurs when [$n=1$ or $g(Y)=1$] and f is unramified.

<u>Lüroth Theorem</u>: If L is a subfield of a pure transcendental ... $k \hookrightarrow L \hookrightarrow k(t)$

Lüroth Theorem: If L is a subfield of a pure transcendental extension $k(t)$ of k, containing k. $k \hookrightarrow L \hookrightarrow k(t)$
then L is also pure transcendental extⁿ.

Pf: Assume $L \neq k$, so that L has transcendence deg 1 over k.
Then L is a function field of a curve Y and the inclusion $L \subset k(t)$ corresponds to a finite morphism $f: \mathbb{P}^1 \to Y$

$\Rightarrow g(Y) = 0 \Rightarrow Y \cong \mathbb{P}^1$. Hence $L \cong k(u)$ for some u. $\square$.