Serre's GAGA

Wednesday, June 10, 2020 6:00 PM

1. Analytic Spaces

$U \subset \mathbb{C}^n$ open subspace

$\mathcal{H}$ sheaf of holomorphic functions on U.

Def^n (Cartan): let $f_1, \dots, f_r$ be holo functions on U,

$\mathcal{I}$ coherent sheaf of ideals generated by f_i on U.

An affine analytic space $(X, \mathcal{H}_X)$ is a locally ringed space whose underlying space $X = \{ y \in U \mid f_1(y) = \dots = f_r(y) = 0 \} \subset U$.

$\mathcal{H}_X := i^{+}\mathcal{H}/\mathcal{I}$ where $i : X \to U$ is the inclusion map.

— Stalk of $\mathcal{H}_{X,x}$?

An analytic space $(X, \mathcal{H}_X)$ locally ringed space, satisfying

(i) $\exists$ open cover $\{V_i\}$ of X s.t. $(V_i, \mathcal{H}_X|_{V_i})$ affine ana. space $\forall i$.

(ii) X is separated (Hausdorff) top. space.

Morphisms are Morphisms of locally ringed spaces.

- Check we can really glue.
- X is smooth, AS $\longleftrightarrow$ Complex manifold.

  $\mathcal{O}_X < \mathcal{H}_X$ sheaf of regular (algebraic) functions on X

X $\mathbb{C}$-scheme of locally finite type $\overset{\text{associate}}{\rightsquigarrow} (X^{an}, \mathcal{O}^{an})$ AS.

$\lambda_X : (X^{an}, \mathcal{O}^{an}) \to (X, \mathcal{O}_X)$

locally R.S

Ex: Show $\Lambda_X : \text{AnSp} \longrightarrow \text{Sets}$

X - $\mathbb{C}$-scheme lft

$\mathcal{X} \longmapsto \text{Hom}_{\mathbb{C}}(\mathcal{X}, X)$
loc. R.S

is representable by the analytic space X^{an}.

and the equivalence $\text{Hom}(-, X^{an}) \cong \Lambda_X$ is induced by λ_X. [Hint: Reduce to $X = \mathbb{A}^1_{\mathbb{C}}$]

- $f : X \to Y$ induces $f^{an} : X^{an} \to Y^{an}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \lambda_X}\uparrow & \text{can} & \downarrow{\scriptstyle \lambda_Y} \end{array}$$

- $f: X \to Y$ induces [illegible]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \lambda_X \uparrow & & \uparrow \lambda_Y \\ X^{an} & \xrightarrow{f^{an}} & Y^{an} \end{array}$$

- $(X \times_Z Y)^{an} \cong X^{an} \times_{Z^{an}} Y^{an}$

- X scheme lft $/ \mathbb{C}$, $P = \{$non-empty, CM, normal, reduced, regular of dim $n\}$

  X has P iff X^{an} has P.

Fact: X is connected (irreducible) iff X^{an} is connected (resp. irre)

- $f: X \to Y$ morphism of $\mathbb{C}$-schemes of locally finite type

  $f^{an}: X^{an} \to Y^{an}$ induced map.

  $P := \{$flat, étale, unramified, open immersions, smooth$\}$

  Then f has P iff f^{an} has P.

  if f finite type $P = \{$surj., dominant, proper, proj., finite$\}$

  f has P iff f^{an} has P.

  [Use: $\mathcal{O}_x^{an}$ is flat over $\mathcal{O}_{X,x}$]

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4. The GAGA Theorem:

Ex: Show that the analytification of a coherent sheaf defined by Kamil (as in Neeman) was nothing but

$\underbrace{\lambda_X^* \mathcal{F}}_{\text{coherent}}$ for any coherent sheaf on X. $\lambda_X: X^{an} \to X$

follows from flatness of λ_X.

$$\mathrm{Coh}\, X \longrightarrow \mathrm{Coh}\, X^{an}$$

$$\mathcal{F} \longmapsto \lambda^* \mathcal{F} = \lambda^{-1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}^{an}$$

$$\lambda^*\mathcal{F} = \mathcal{F}^{an}$$

Prop^n: The functor $\mathcal{F} \to \mathcal{F}^{an}$ is exact, faithful and conservative.

$f \mapsto F(f)$, $F(f)$ iso $\Rightarrow f$ is iso.

Pf: - Exactness: λ^{-1} exact $+$ $\mathcal{O}^{an}$ flat over $\mathcal{O}_X$.

Since $\mathcal{O}_{X,x} \to \mathcal{O}_X^{an}$ faithfully flat

for each $x \in X^{an}$, $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_x^{an} = 0$ iff $\mathcal{F}_x = 0$

Since $X(\mathbb{C})$ dense in X. $\Rightarrow \mathcal{F} = 0$ iff $\mathcal{F}_x = 0 \ \forall x \in X(\mathbb{C})$.

$\square$

λ^* has left adjoint λ_*

$$\mathcal{F} \longrightarrow \lambda_* \lambda^* \mathcal{F} = \lambda_* \mathcal{F}^{an}$$

[Ex: $\varphi : X \to S$ morphism of schemes $\mathcal{F} \in Coh(X)$ $\quad$ $\mathcal{F}$ is flat/S iff
$\varphi^{an} : X^{an} \to S^{an}$ $\quad \mathcal{F}^{an} \in Coh(X^{an})$ $\quad$ $\mathcal{F}^{an}$ is flat/S]

$\mathcal{F} \in Mod(\mathcal{O}_X)$

$$\begin{tikzcd}
R^\bullet f_* \mathcal{F} \arrow[r] & R^\bullet f_*(\lambda_* \mathcal{F}^{an}) \arrow[r, "\sim"] & R^\bullet(\lambda \circ f)_* \mathcal{F}^{an} \arrow[d, "\wr"] \\
& & R^\bullet(f^{an} \circ \lambda_Y)_* \mathcal{F}^{an} \arrow[d] \\
& & \lambda_{Y*} R^\bullet f_*^{an} \mathcal{F}^{an}
\end{tikzcd}$$

how do we get this map.

(λ^*, λ_*) adjoint pair,

$$\lambda^* \lambda_* \longrightarrow id_{Mod(\mathcal{O}_Y^{an})}$$

$$\lambda_Y^* \lambda_{Y*} f_*^{an} I^\bullet \longrightarrow f_*^{an} I^\bullet \qquad I^\bullet \to \mathcal{F}^{an} \text{ inj. reso.}$$

induces a natural morphism of right derived functors:

$$\lambda_Y^* R^\bullet \lambda_{Y*} f_*^{an} \mathcal{F}^{an}$$

$\wr\wr \leftarrow$ exactness of λ_Y^*

$$R^\bullet \lambda_Y^* \lambda_{Y*} f_*^{an} \mathcal{F}^{an}$$

$\downarrow \qquad \leftarrow$ take adjoint

$$R^\bullet f_* \mathcal{F}^{an}$$

Thus we have an induced morphism

$$\theta^\bullet : (R^\bullet f_* \mathcal{F})^{an} \longrightarrow R^\bullet f_*^{an} \mathcal{F}^{an} \text{ of } \mathcal{O}_Y^{an}\text{-modules}$$

$f : X \to Y$ is a proper morphism

$\theta : (R f_* \mathcal{F})^{an} \to R f_*^{an} (\mathcal{F}^{an})$

Now assume $f : X \to Y$ is a proper morphism

$$f_* : Coh(X) \to Coh(Y)$$

$$R^p f_* \mathcal{F} \cong \check{H}^p (X, \mathcal{F}, f_*) = \varinjlim_{\mathcal{U}} H^p (f_* \mathcal{C}^\cdot (\mathcal{U}, \mathcal{F}))$$

$\mathcal{U}$ open cover of X, colimit is over refinments.

Similarly, on analytic side:

$\tilde{f} : \mathcal{X} \to \mathcal{Y}$ proper morphism of AnSp, $\mathcal{F} \in Mod(\mathcal{O}_{\mathcal{X}}^{an})$ coherent.

then $R^p \tilde{f}_* \mathcal{F} \cong \check{H}^p (\mathcal{X}, \mathcal{F}, \hat{f}_*) = \varinjlim_{\mathcal{U}} H^p (\tilde{f}_* (\mathcal{C}^\cdot(\mathcal{U}, \mathcal{F}))$

Theorem: let $f : X \to Y$ be a proper morphism of $\mathbb{C}$-schemes locally of finite type and $\mathcal{F} \in Coh(X)$ Then for any $p \geq 0$, the morphism

$$\theta^p : (R^p f_* \mathcal{F})^{an} \to R^p f_*^{an} (\mathcal{F}^{an})$$

is an isomorphism.

In case $Y = Spec(\mathbb{C})$

$$H^p(X, \mathcal{F}) \cong H^p(X^{an}, \mathcal{F}^{an}), \quad p \geq 0$$

Remark: The above result fails without properness assumption:

Eg: On the affine line; $\mathbb{A}^1_{\mathbb{C}}$ there are many analytic functions that are not algebraic (eg: $\sin x$)

Hence $H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \ncong H^0(\mathbb{C}, \mathcal{O}^{an})$

The main GAGA Thm:

let X be a proper $\mathbb{C}$ scheme and $\lambda : X^{an} \to X$ be the

Let X be a proper ~~C~~ scheme and $\lambda : X^{an} \to X$ be the canonical morphism. Then the functor

$$\lambda^* : Coh_X \longrightarrow Coh_{X^{an}}$$
$$\mathcal{F} \longmapsto \mathcal{F}^{an}$$

is an equivalence of categories.

Chow lemma: Let X be a proper scheme. Then any closed analytic subspace $\mathcal{Y}$ of X^{an} is analytification of some closed subscheme $Y \subset X$, i.e. $\mathcal{Y}$ is algebraic.

Pf: Ideal sheaves.

Proof of fully faithfulness:

$\mathcal{F}, \mathcal{G} \in Coh_X$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent sheaf.

$$Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \quad [\text{Thm above}]$$
$$\cong H^0(X^{an}, \underline{\mathcal{H}om(\mathcal{F}, \mathcal{G})^{an}})$$

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})^{an} \cong \mathcal{H}om_{\mathcal{O}^{an}}(\mathcal{F}^{an}, \mathcal{G}^{an})$$

$\Rightarrow \lambda^*$ is fully faithful. $\square$.