EXERCISE SHEET 2: ALGEBRAIC CURVES AND MODULI SPACES

TANYA KAUSHAL SRIVASTAVA

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1. More on Divisors

(1) (Liu Proposition 7.2.9) Let $X = \mathbb{P}^n_k$ be a projective space over a field k, with $n \geq 1$. Then there exists a group homomorphism $\delta : Z^1(X) \to \mathbb{Z}$ which induces an isomorphism $Cl(X) \to \mathbb{Z}$. Moreover, any hyperplane in X is a generator of $Cl(X)$.

(2) (Liu Theorem 7.2.14, normality assumption is necessary, Example 7.2.15) Show that if X is not normal, then the homomorphism $Div(X) \to Z^1(X)$ is not necessarily injective, where $Div(X)$ is the group of Cartier divisors and $Z^1(X)$ the group of Weil divisors.

(3) (Liu Proposition 7.2.16) Let X be a Noetherian regular integral scheme. Then the homomorphisms

$$Div(X) \to Z^1(X),\ CaCl(X) \to Cl(X)$$

are isomorphism.

(4) (Liu Exercise 7.2.4) Let X be a scheme. Let $D \in Div(X)$ be a Cartier divisor.
   (a) If D is effective, show that Supp $D = V(\mathcal{O}_X(-D))$.
   (b) Let us suppose X is Noetherian. Show that Supp $[D] \subset$ Supp D, and we have an equality if X is regular or D is effective.
   (c) Compare Supp D and Supp $[D]$ in the example you constructed for exercise (2) above.
   (d) Let $f : X \to Y$ be a morphism such that either f flat or X is reduced, having only finitely many irreducible components and each of these dominates an irreducible component of Y. Let E be an effective Cartier divisor on Y. Show that Supp $f^*E = f^{-1}(\text{Supp } [E])$.

2. Cohomology

We will do cohomology of projective spaces in class.

(1) (Euler characteristic, Hartshorne III.5.1) Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the **Euler characteristic** of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum (-1)^i dim_k H^i(X, \mathscr{F}).$$

If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

(2) (Hilbert Polynomial, Hartshorne Ex III 5.2) Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in$

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$\mathbb{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the **Hilbert polynomial** of $\mathscr{F}$ with respect to the sheaf $\mathcal{O}_X(1)$.

(3) (Arithmetic Genus, Hartshorne Ex III 5.3) Let X be a projective scheme of dimension r over a field k. We define the **arithmetic genus** p_a of X as
$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$
Note that it depends only on X and not on any projective embedding of X. If X is integral and k is algebraically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that
$$p_a(X) = \sum_{i=0}^{r-1}(-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$
In particular, if X is an integral curve, we have $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$.

(4) (Computing Arithmetic genus, Hartshorne Ex I 7.2)
   (a) Note that $p_a(\mathbb{P}^n) = 0$.
   (b) If X is an irreducible curve in $\mathbb{P}^2$ of degree d, show that $p_a(X) = 1/2(d-1)(d-2)$.
   (c) If X (resp. Y) is a projective variety of dimension r (resp. s). Show that
$$p_a(X \times Y) = p_a(X)p_a(Y) + (-1)^r p_a(X) + (-1)^s p_a(Y).$$

(5) (Geometric Genus, Hartshorne Ex. II 8.3) For a smooth projective variety X over k of dimension n, we define the **geometric genus** to be $p_g = \dim_k H^0(X, \omega_{X/k})$, where $\omega_{X/k} := \wedge^n \Omega_{X/k}$, the n-th exterior power of the sheaf of differentials. This is an invertible sheaf. Note that in case of smooth projective curves the arithmetic and geometric genus coincide. Let Y be a nonsingular cubic curve in $\mathbb{P}^2$ and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$. This shows that arithmetic and geometric genus of a non-singular projective variety may be different.