EXERCISE SHEET 1: ALGEBRAIC CURVES AND MODULI SPACES

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1. Exercises on Curves

Curves in the following exercises are assumed to be irreducible.

(1) (Hartshorne I Ex. 5.3) Multiplicity: Let $Y \subset \mathbb{A}^2$ be a curve defined by equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \ldots f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the **multiplicity** of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y$ iff $\mu_P(Y) > 0$.) The linear factors of f_r are called the tangent directions at P.
   - (a) Show that $\mu_P(Y) = 1$ iff P is a nonsingular point of Y.
   - (b) Find the multiplicity of each of the singular points in $x^2 - x^4 - y^4 = 0$, $xy = x^6 = y^6$, $x^3 = y^2 + x^4 + y^4$, $x^2y + xy^2 = x^4 + y^4$.

(2) (Hartshorne I Ex. 5.4) Intersection multiplicity: If $Y, Z \subset \mathbb{A}^2$ are two distinct curves, given by equations $f = 0$, $g = 0$ and if $P \in Y \cap Z$, define the **intersection multiplicity** $(Y.Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f, g)$.
   - (a) Show that $(Y.Z)_P$ is finite, and $(Y.Z)_P \geq \mu_P(Y)\mu_P(Z)$.
   - (b) If $P \in Y$, show that almost all lines L through P (i.e., all but a finite number), $(L.Y)_P = \mu_P(Y)$.
   - (c) If Y is a curve of degree d in $\mathbb{P}^2$ (i.e., defined by an equation of degree d), and if L is a line in $\mathbb{P}^2$, $L \neq Y$, then define $(L.Y) := \sum(L.Y)_P$ taken over all points $P \in L \cap Y$, where $(L.Y)_P$ is defined using a suitable affine cover. Show that $(L.Y) = d$.

(3) (Hartshorne I Ex. 5.11) Elliptic Quartic Curve in $\mathbb{P}^3$: Let $Y := Z(x^2 - xz - yw, yz - xw - zw)$ be the zero set in $\mathbb{P}^3$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$, and let φ denote the projection from P to the plane $w = 0$.
   - (a) Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1, 0, -1)$.
   - (b) Show that Y is an irreducible nonsingular curve.

   The curve Y is called the elliptic quartic curve in $\mathbb{P}^3$. This is an example of a complete intersection.

2. Associated points on a scheme

(1) (Liu Ex. 2.1.4) Let A be a ring.
   - (a) Let $\mathfrak{p}$ be a minimal prime ideal of A. Show that $\mathfrak{p}A_{\mathfrak{p}}$ is nilpotent. Deduce from this that every element of $\mathfrak{p}$ is a zero divisor in A.
   - (b) Show that if A is reduced, then any zero divisor in A is an element of a minimal prime ideal. Show that this is false if A is not reduced.

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(2) (Liu Remark 7.1.10) If X has no embedded points, then $Ass(\mathcal{O}_X) \subset U$ if and only if U is everywhere dense in X.
(3) (Liu Remark 7.1.14) If X is locally Noetherian, or if it is reduced with only finitely many irreducible components, then for any affine open subset U of X we have $\mathcal{K}_X(U) = \mathcal{K}'_X(U) = Frac(\mathcal{O}_X(U))$.
(4) Show that generic points of X are associated points of X.
(5) (Liu Ex. 7.1.2) Let X be a locally Noetherian scheme without embedded points. Show that X is reduced if and only if it is reduced at the generic points.
(6) (Liu Ex. 7.1.3) Let X be a locally Noetherian scheme. we suppose that there exists a unique point $x \in X$ such that $\mathcal{O}_{X,x}$ is not reduced. Show that $x \in Ass(\mathcal{O}_X)$.

3. Cartier Divisor

(1) (Liu Ex 7.1.13) Let $\mathcal{L}$ be an invertible sheaf on an integral scheme X. Let $s \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X)$ be a non-zero rational section of $\mathcal{L}$.
   (a) Let $\{U_i\}_i$ be a covering of X such that $\mathcal{L}|_{U_i}$ is free and generated by the elements e_i. Show that there exist $f_i \in K(X)^*$ (Recall that $K(X) := \mathcal{O}_{X,\xi}$ for the unique generic point ξ of X) such that $s|_{U_i} = e_i f_i$. Show that $\{(U_i, f_i)\}_i$ defines a Cartier divisor on X. We denote it by $div(s)$. Show that $\mathcal{O}_X(div(s)) = \mathcal{L}$.
   (b) If $\mathcal{L} = \mathcal{O}_X$, show that $div(s)$ is the principal Cartier divisor associated to s.
   (c) Show that $div(s) \geq 0$ iff $s \in \Gamma(X, \mathcal{L})$.
   (d) Let $D \in Div(X)$. For any open subset U of X, show that
$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}_X^*(U) | div(f) + D|_U \geq 0\} \cup \{0\}.$$