EXERCISE SHEET 4: ALGEBRAIC CURVES AND RIEMANN SURFACES

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1. Algebraic world: exercises from Hartshorne

(1) (4 pts) Locate the singular points and draw the following curves in $\mathbb{A}^2$ (assume char $k \neq 2$)
 (a) $x^2 = x^4 + y^4$
 (b) $xy = x^6 + y^6$
 (c) $x^3 = y^2 + x^4 + y^4$
 (d) $x^2y + xy^2 = x^4 + y^4$.

(2) (6 pts) Multiplicity: Let $Y \subset \mathbb{A}^2$ be a curve defined by equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \ldots f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the **multiplicity** of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y$ iff $\mu_P(Y) > 0$.) The linear factors of f_r are called the tangent directions at P.
 (a) Show that $\mu_P(Y) = 1$ iff P is a nonsingular point of Y.
 (b) Find the multiplicity of each of the singular points in the exercise above.

(3) (6 pts) Intersection multiplicity: If $Y, Z \subset \mathbb{A}^2$ are two distinct curves, given by equations $f = 0$, $g = 0$ and if $P \in Y \cap Z$, define the **intersection multiplicity** $(Y.Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_{P,\mathbb{A}^2}$-module $\mathcal{O}_{P,\mathbb{A}^2}/(f, g)$.
 (a) Show that $(Y.Z)_P$ is finite, and $(Y.Z)_P \geq \mu_P(Y)\mu_P(Z)$.
 (b) If $P \in Y$, show that almost all lines L through P (i.e., all but a finite number), $(L.Y)_P = \mu_P(Y)$.
 (c) If Y is a curve of degree d in $\mathbb{P}^2$ (i.e., defined by an equation of degree d), and if L is a line in $\mathbb{P}^2$, $L \neq Y$, then define $(L.Y) := \sum (L.Y)_P$ taken over all points $P \in L \cap Y$, where $(L.Y)_P$ is defined using a suitable affine cover. Show that $(L.Y) = d$.

2. World of Riemann Surfaces-from Miranda's book

Let X be a Riemann Surface, let p be a point of X, and let f be a complex valued function defined in a neighborhood W of p.

Definition 1. We say that f is **holomorphic at** p if there exists a chart $\phi : U \to V$ with $p \in U$, such that the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. We say f is **holomorphic in** W if it is holomorphic at every point of W.

(1) (2 pts) Show that f is holomorphic at p iff for every chart $\phi : U \to V$ with $p \in U$, the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. Moreover, note that this result implies that f is holomorphic in W if and only if there exist a set

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of charts $\{\phi_i : U_i \to V_i\}$ with $W \subseteq \cup_i U_i$, such that $f \circ \phi_i^{-1}$ is holomorphic on $\phi_i(W \cap U_i)$ for each i. Further, note that if f is holomorphic at p, then f is holomorphic in a neighborhood of p.

(2) (3 pts) Let f be a complex valued function on the Riemann sphere, defined in a neighborhood of ∞, show that f is holomorphic at ∞ iff $f(1/z)$ is holomorphic at $z = 0$. In particular, if f is a rational function $f(z) = p(z)/q(z)$, then f is holomorphic at ∞ iff $\deg p \leq \deg q$.

(3) (4 pts) Sketch the proofs of examples II.1.7 to II.1.10 in Miranda's book.