Defn: X be a projective curve over a field k, $\mathcal{L}$ invertible sheaf on X. We define the degree of $\mathcal{L}$ to be the integer
$$\deg \mathcal{L} := \chi(\mathcal{L}) - \chi(\mathcal{O}_X).$$

Lemma: X proj curve$/k$

a) If $\mathcal{L} \simeq \mathcal{O}_X(D)$ for some $D \in \mathrm{Div}(X)$, then $\deg \mathcal{L} = \deg D$

b) $\mathcal{L} \longmapsto \deg \mathcal{L}$ is a group homo $\mathrm{Pic}(X) \to \mathbb{Z}$

Pf: a) Riemann $\deg D = \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X)$

b) $\mathrm{CaCl}(X) \simeq \mathrm{Pic}(X)$, $\deg \mathcal{O}_X = 0$.

Defn: lci : Y locally noetherian scheme $f: X \to Y$ morphism of finite type. Then we say that f is a local complete intersection (lci) at x if $\exists$ $x \in U$ open nbhd and a commutative diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ U \arrow[ur, hook, "i"] \arrow[r, "f|_U"'] & Y \end{tikzcd}$$

i regular immersion, g is a smooth morphism.

We say f is lci if it is lci at every point of X.

Examples: $f: X \to Y$ is regular immersion / smooth.

Corollary: Let X be an lci proj. curve$/k$, with (arithmetic) genus p_a. Then

a) $\deg \omega_{X/k} = 2(p_a - 1)$

b) $\dim_k H^0(X, \omega_{X/k}) = p_a$ if X is geom. connected & geo. reduced.

Pf: Pick a D s.t $\mathcal{O}_X(D) \simeq \omega_{X/k}$
$$\deg \omega_{X/k} = \chi(\omega_{X/k}) - \chi(\mathcal{O}_X) \quad p_a(X) - 2$$
$$= \dim H^0(X, \omega_{X/k}) - \dim H^1(X, \omega_{X/k})$$
$$= \underset{\parallel}{} H^1(X, \mathcal{O}_X)^\vee \qquad H^0(X, \mathcal{O}_X)^\vee$$
$$= -\chi(\mathcal{O}_X) + p_a(X) - 1$$
$$= 2(p_a(X) - 1).$$

b) Use $H^0(X, \omega_{X/k}) = p_a$ + RR.

Defn: X projective curve lci over a field k. Any Cartier Divisor K on X such that $\mathcal{O}_X(K) \simeq \omega_{X/k}$ is called a canonical divisor.

- Such a divisor always exists.
- $K_{X/k}$ (denote) (only defined upto linear equivalence).

Remark: R-R theorem can be restated as $D \in \mathrm{Div}(X)$
$$\ell(D) - \ell(K - D) = \deg D + 1 - p_a(X).$$

- If X is integral and $\deg D > 2p_a(X) - 2 = \deg K$ then
$$\ell(D) = \deg D + 1 - p_a(X)$$

- X be a lci projective curve/k s.t. $\omega_X \simeq \mathcal{O}_X \Rightarrow p_a(X) = 1$

$\left[\begin{array}{c} \deg(K-D) < 0 \\ \Downarrow \\ \ell(K-D) = 0 \end{array} \right]$

- Conversely, X be an integral projective lci/k with $p_a(X) = 1$, then $\omega_{X/k} \simeq \mathcal{O}_X$.
- X integral projective lci of genus $p_a = 0$.

$$D \in \mathrm{Div}(X). \quad \ell(D) = \begin{cases} \deg D + 1, & D \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

Defn: X scheme, $X_1, \ldots, X_n$ be irreducible components with reduced closed subscheme st.

$X' = \bigsqcup_{1 \leq i \leq n} X_i'$, where X_i' is the normalization of X_i.

$\hookrightarrow$ normalization of X.

By construction, X' is endowed with a surjective integral morphism $\pi : X' \longrightarrow X$ (Moreover, π is finite) X red.

- Note $X'_{red} = X'$.

let X be a reduced curve/k. $\pi : X' \to X$ normalization (finite)

We have an exact sequence of sheaves on X (coherent):

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_{X'} \longrightarrow S \longrightarrow 0 \qquad (1)$$

- $\mathrm{Supp}(S)$ is a closed set not containing any generic point of X.

$\Rightarrow$ $\mathrm{Supp}\, S$ is a finite set, consisting of singular points

$\Rightarrow$ S is a skyscrapper sheaf. (direct sum)

For any $x \in X$, we have $S_x = \mathcal{O}'_{X,x} / \mathcal{O}_{X,x}$

$\mathcal{O}'_{X,x} :=$ integral closure of $\mathcal{O}_{X,x}$ in $\mathrm{Frac}(\mathcal{O}_{X,x})$

[normalization commutes with localization]

let $\delta_x := \mathrm{length}_{\mathcal{O}_{X,x}} S_x = [k(x):k]^{-1} \dim S_x$. (exercise)

Then $\delta_x = 0$ iff x is normal in X (hence regular)

Propn: X reduced projective curve over a field k, $X_1, \ldots, X_n$ be the irreducible components of X. Then

$$p_a(X) + n - 1 = \sum_{1 \leq i \leq n} p_a(X_i') + \sum_{x \in X} [k(x):k]\, \delta_x .$$

X_i' is normalization of X_i.

Proof: $0 \to \mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_{X'} \longrightarrow S \longrightarrow 0$

$$\Rightarrow \chi(\pi_* \mathcal{O}_{X'}) = \chi(\mathcal{O}_X) + \chi(S)$$

As π is finite

$$\chi(\pi_* \mathcal{O}_{X'}) = \chi(\mathcal{O}_{X'}) = \sum_{1 \leq i \leq n} \chi(\mathcal{O}_{X_i'})$$
$$= n - \sum_{1 \leq i \leq n} p_a(X_i')$$

Moreover, since S is skyscrapper

$$\chi(S) = \dim_k H^0(X, S) = \sum_x \dim_k S_x = \sum_x [k(x):k]\, \delta_x$$

$\square$

Prop^n: X geo integral projective curve$/k$ of $p_a \leq 0$.
Then we have the following properties :-
(a) The curve X is a smooth conic$/k$
(b) we have $X \cong \mathbb{P}^1_k$ iff $X(k) \neq \emptyset$.

Proof:
X' normalization of $X_{\bar{k}}$
$H^0(X', \mathcal{O}_{X'}) = \bar{k}$
$\Rightarrow p_a(X') \geq 0 \quad \Rightarrow p_a(X') = 0$
$\Rightarrow X_{\bar{k}} = X'$ (i.e $X_{\bar{k}}$ is normal)
$\Rightarrow X$ is smooth over k.

b) suppose that there exists an $x_1 \in X(k)$
Then $l(x_1) = 2$
$\Rightarrow X \cong \mathbb{P}^1_k$ + $\mathcal{O}_X(x_1)$ is very ample.

a) Let K be a canonical divisor of X, and let $D = -K$
Then $\deg D = 2$ and $l(D) = 3$. $2(p_a - 1)$

Claim : $\mathcal{O}_X(D)$ is very ample.
[Assume the claim, then $\mathcal{O}_X(D)$ induces a closed immersion from X to $\mathbb{P}^2_k$ and image is a conic, by the genus formula $(d-1)(d-2)/2$].

Proof of claim: wlg base change to alg. closure
Let $x_1 \in X(k)$

Then $l(D - 2x_1) \geq 1$ and therefore $D \sim 2x_1$
[use $\deg(D - 2x_1) = 0$ but $l(D - 2x_1) \neq 0$)
$\Rightarrow D - 2x_1 \sim 0$
$\Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(x_1)^{\otimes 2}$
$\uparrow$ very ample
$\Rightarrow \mathcal{O}_X(D)$ is very ample. $\square$

Remark: Proof also shows that on a smooth, irreducible proj. curve of genus 0, every divisor of deg 1 or 2 is very ample.
[$\Rightarrow$ Every divisor of strictly +ve degree is very ample].

Lemma 1: Let X be a proj. curve$/k$ and let $D \in \mathrm{Div}_+(X)$ with support in the regular locus of X. Then $\mathcal{O}_X(D)$ is generated

Lemma 1. Let X be a proj. curve/k [illegible] support in the regular locus of X. Then $\mathcal{O}_X(D)$ is generated by its global sections iff for any $x \in \operatorname{Supp} D$ we have $\ell(D-x) < \ell(D)$.

Proof: As $D \geq 0$ $\mathcal{O}_X \subset \mathcal{O}_X(D)$

hence $1 \in H^0(X, \mathcal{O}_X(D))$

$\Rightarrow$ $\mathcal{O}_X(D)$ is generated by it global sections at every point $x \notin \operatorname{Supp}(D)$.

Let $x \in \operatorname{Supp}(D)$. Then $D - x \leq D$

$\rightarrow \ell(D-x) \leq \ell(D)$

$$\mathcal{O}_X(D-x)_x = m_x \mathcal{O}_X(D)_x \quad \text{[Exercise]}$$

If $\ell(D-x) < \ell(D)$, then there exists $s \in L(D) \setminus L(D-x)$ we then have $s_x \notin \mathcal{O}_X(D-x)_x = m_x \mathcal{O}_X(D)_x$

$\Rightarrow$ s_x is a generator of $\mathcal{O}_X(D)_x$ and $\mathcal{O}_X(D)$ is generated by its global sections at x.

Conversely, suppose $\mathcal{O}_X(D)$ is generated by its global sections at x. Let $s \in L(D)$ be such that s_x is a generator of $\mathcal{O}_X(D)_x$. Then $s_x \notin L(D-x)_x$ and hence $s \in L(D-x)$.

$\Rightarrow \ell(D-x) < \ell(D)$. $\square$.

Lemma 2: Let X be a connected smooth proj. curve over $k = \bar{k}$. Let $D \in \operatorname{Div}_+(X)$ such that for any pair of (not necessarily distinct) points $p, q \in X(k)$, we have

$$\ell(D-p-q) < \ell(D-p) < \ell(D).$$

Then $\mathcal{O}_X(D)$ is very ample.

Sketch: From the lemma 1 above, $\mathcal{O}_X(D)$ is generated by its

Sketch: From the lemma 1 above, $\mathcal{O}_X(D)$ is generated by its global sections.

let $\{s_0, \dots, s_n\}$ be a basis of $L(D)$ and $f: X \to \mathbb{P}^n_k$ be the associated morphism.

Claim: f is injective

p, q be two distinct closed points.

Then $\exists\, s \in L(D-p) \setminus L(D-p-q)$.

$\Rightarrow$ We have $s_p \in m_p \mathcal{O}_X(D)_p$ while s_q is a generator of $\mathcal{O}_X(D)_q$.

Write $s = \sum_{0 \le i \le n} \lambda_i s_i \quad \lambda_i \in k$.

and $f(p) = (p_0, \dots, p_n)$, $f(q) = (q_0, \dots, q_n)$.

Fix a basis e of $\mathcal{O}_X(D)_p$.

then $(s_i)_p \in p_i e + m_p^2$

$\Rightarrow \sum_i \lambda_i p_i = 0$

(check)

But $\sum \lambda_i q_i \neq 0$. $\Rightarrow f(p) \neq f(q)$.

Exercise: Show $T_{f,p}$ (the tangent map) is injective for every $p \in X(k)$.

$\square$

Prop^n: Let X be a smooth, geo. connected projective curve over a field k of genus g. Let $\mathcal{L}$ be an invertible sheaf on X.

(a) If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is generated by its global sections

(b) If $\deg \mathcal{L} \geq 2g+1$, then $\mathcal{L}$ is very ample

Proof: wlog assume k is alg. closed
let D be a Cartier divisor such that $\mathcal{O}_X(D) \simeq \mathcal{L}$.
First note that if $\deg D \geq 2g$ (resp. $\deg D \geq 2g+1$)
then $\ell(D-E) = \ell(D) - \deg E$ for $E \in \mathrm{Div}_X(k)$
s.t. $\deg E \leq 1$
(resp ≤ 2)

$$\ell(D-E) = \deg(D-E) + 1 - g \quad [\underbrace{\deg(D-E) > 2g-2}_{\text{Remark before}}].$$
$$= \deg D - \deg E + 1 - g$$
$$= \ell(D) - \deg E.$$

In particular $\ell(D) \neq 0$. $(\ell(D) \geq \deg D + 1 - g)$

Hence by linear equivalence, we can reduce to $D \geq 0$.
Then lemma 1 $\Rightarrow$ global gene
2 $\Rightarrow$ very ample. $\square$

Corollary: X smooth proj. connected curve $/k$ of genus 1, suppose $\exists\ o \in X(k)$. Then X is an elliptic curve and $\mathcal{O}_X(3o)$ is very ample.

Proof: Next time.