Lecture 9: Riemann-Roch for Curves I and Dualizing sheaf

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Thm (Riemann): X proj. curve $/k$. $D \in \mathrm{Div}(X)$. Then

$$\chi(\mathcal{O}_X(D)) = \deg D + \underset{\underset{\chi(X)}{\parallel}}{\chi(\mathcal{O}_X)}$$

Proof: Recall $D \in \mathrm{Div}(X)$

$D = E \setminus F$ with E, F effective Cartier divisors (non-zero)

We have an exact sequence

$$0 \to \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_F \to 0$$

Tensor with $\mathcal{O}_X(E)$ over $\mathcal{O}_X$

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(E) \to \mathcal{O}_X(E)|_F \to 0$$

F is finite scheme $\Rightarrow \mathcal{O}_X(E)|_F \simeq \mathcal{O}_F$

Moreover, $H^q(F, \mathcal{O}_F) = 0$ for $q \geq 1$ (Grothendieck Vanishing)

Recall $\deg D = \dim_k H^0(D, \mathcal{O}_D)$

then, additivity of Euler characteristic

$$\Rightarrow \chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(E)) - \deg F$$

Apply to $D = 0$, we obtain $\chi(\mathcal{O}_X(E)) = \chi(\mathcal{O}_X) + \deg E$

$$\Rightarrow \chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \underset{\parallel}{\deg E - \deg F}$$
$$= \chi(\mathcal{O}_X) + \deg D. \qquad \square$$

Corollary: X proj curve$/k$. Let $\mathrm{div}(f)$ principal Cartier divisor on X.

Then $\deg(\mathrm{div}(f)) = 0$

Proof: $D = \mathrm{div}(f)$, $\mathcal{O}_X(D) \simeq \mathcal{O}_X \Rightarrow \deg(\mathrm{div}(f)) = 0$. $\square$

Defn: X proj curve$/k$

Arithmetic genus of X: $p_a(X) := 1 - \chi_k(\mathcal{O}_X)$

If X is geo. connected, geo. reduced,

$H^0(X, \mathcal{O}_X) = k$.

$\Rightarrow p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$.

Geometric genus $p_g(Y) := \dim_k H^0(Y, \omega_{Y/k})$ — smooth proj. variety Y

$\omega_{Y/k}$: sheaf of differentials.

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sheaf of differentials.

$\dim Y = n$ $\quad \omega_{Y/k} = \Lambda^n \Omega_{Y/k}$

Y is a curve

$g(Y)$ ← denote the geo. genus.

$D \in \mathrm{Div}(X)$, $\quad L(D) := H^0(X, \mathcal{O}_X(D))$

$\ell(D) = \dim_k L(D)$.

Remark: X normal proj. curve $/k$

$Z = \sum_x n_x [x]$ any 0-cycle

$D \in \mathrm{Div}(X)$ such that $[D] = Z$

$L(D) = \{ f \in K(X)^* \mid \mathrm{mult}_x(f) + n_x \geq 0, \forall x \} \cup \{0\}$

closed in X.

X integral,

$L(D) = \{ f \in K(X)^* \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\}$.

Example: (i) $X = \mathbb{P}^1_k$ $\quad p_a = p_g = 0$

(ii) X proj. plane curve$/k$ defined by F homogeneous polynomial of degree $n \geq 1$

Then $H^0(X, \mathcal{O}_X) = k$, $\quad p_a(X) = \dfrac{(n-1)(n-2)}{2}$

In particular elliptic curves have genus 1.

k be field, we define elliptic curves over k to be a smooth projective curve E/k, isomorphic to a closed subvariety of $\mathbb{P}^2_k$ defined by

$$F(u,v,w) = v^2 w + (a_1 u + a_3 w) vw - (u^3 + a_2 u^2 w + a_4 u w^2 + a_6 w^3)$$

with the privileged rational pt. $O = (0,1,0)$.

Corollary: X proj $/k$. $D \in \mathrm{Div}(X)$. Then

$$\ell(D) \geq \deg D + 1 - p_a(X).$$

Corollary: X normal proj. curve $/k$. Then $X \cong \mathbb{P}^1_k$ iff there

$$l(D) \geq \deg D + 1 - p_a(X).$$

Corollary: X normal proj curve $/k$. Then $X \cong \mathbb{P}^1_k$ iff there exists a Cartier divisor D such that $\deg D = 1$ and $l(D) \geq 2$. Moreover, for such a D, the sheaf $\mathcal{O}_X(D)$ is very ample.

Pf: $\boxed{\Leftarrow}$ Suppose D as above exists. $g \in L(D) = H^0(X, \mathcal{O}_X(D))$

Then $D \sim \underbrace{\mathrm{div}(g)}_{\text{principal Cartier divisor}} + D \geq 0$.

Hence we may suppose $D \geq 0$.

$\Rightarrow$ D is the divisor associated to a rational point $x_1 \in X(k)$.

$\Rightarrow$ In particular, $H^0(X, \mathcal{O}_X) = k$

(canonical morphism $\underset{\text{field}}{H^0(X, \mathcal{O}_X)} \underset{\text{non-zero}}{\to} k(x_1) = k$),

Let $f \in L(D) \setminus k$, then

$\mathrm{div}(f) + D$ is an effective Cartier divisor distinct from D

It follows that $(f) = [x_0] - [x_1]$ $x_0 \in X(k) \setminus \{x_1\}$

Such an f induces an isomorphism of X to $\mathbb{P}^1$.

$\Rightarrow X \cong \mathbb{P}^1$.

$\boxed{\Rightarrow}$ $X = \mathbb{P}^1_k$ $D \in \mathrm{Div}(X)$

$$l(D) = \begin{cases} \deg D + 1 & \text{if } \deg D \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so pick any k-rational point on $\mathbb{P}^1_k$ as D

$\mathcal{O}_X(D) \simeq \mathcal{O}_X(n)$ $n \in \mathbb{Z}$.

$n = 1$ $D = $ rat. pt. ∎

Propn: X proj. $/k$

(a) Let $D' \leq D$ be Cartier divisors on X such that

$$\dim_k H^0(X, \mathcal{O}_X) = l(0) \leq l(D) \text{ for } D \geq 0.$$

Then we have

$$l(D') \leq l(D) \leq l(D') + \deg(D - D')$$

Then we have
$$\ell(D') \leq \ell(D) \leq \ell(D') + \deg(D-D')$$
In particular if $D \geq 0$, then $\ell(D) \leq \deg D + \dim_k H^0(X, \mathcal{O}_X)$

(b) Let us suppose X is integral. If $\deg D = 0$, then $\ell(D) \neq 0$ iff $D \sim 0$. If $\deg D < 0$, then $\ell(D) = 0$.

(c) X integral, $\mathcal{L}$ invertible sheaf on X. Then $\mathcal{L} \simeq \mathcal{O}_X$ iff $\deg \mathcal{L} = 0$ and $H^0(X, \mathcal{L}) \neq 0$.

Pf: a) $D' \leq D$ $(\Leftrightarrow D - D' \geq 0)$
$$\Rightarrow \mathcal{O}_X(D') \subseteq \mathcal{O}_X(D)$$
$$\Rightarrow \ell(D') \leq \ell(D)$$
Suppose $D' < D$. Write $D = D' + E$ with E non-zero eff. div

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0 \quad \otimes \mathcal{O}_X(D)$$
$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)|_E \to 0.$$
E finite scheme
$$\Rightarrow \mathcal{O}_X(D)|_E \simeq \mathcal{O}_E$$
$$0 \to L(D') \to L(D) \to H^0(E, \mathcal{O}_E)$$
Thus $\ell(D) \leq \ell(D') + \deg(D-D')$ $\quad \deg_k E$.
$\square$.

b) suppose $\ell(D) \neq 0$. Let $f \in L(D) \setminus \{0\}$.
Then $D \sim \operatorname{div}(f) + D \geq 0$
$\Rightarrow \deg D \geq 0$ and

if $\deg D = 0 \Rightarrow \mathrm{div}(f) + D = 0 \Rightarrow D \sim 0$. $\square$

c) Suppose $\deg \mathcal{L} = 0$ and $s \neq 0 \in H^0(X, \mathcal{L})$

Then $\mathcal{L}' := \mathcal{L} \otimes (s\mathcal{O}_X)^\vee$ is an invertible sheaf containing $\mathcal{O}_X$.

$\mathcal{L}' \subset K_{X/k}$ (constant sheaf, X integral)

$\Rightarrow \exists$ a unique effective Cartier divisor D ass. to $\mathcal{L}$

As $\mathcal{L} \simeq \mathcal{L}'$, we have $\deg D = \deg \mathcal{L} = 0$

$\Rightarrow D = 0$; hence $\mathcal{L} \simeq \mathcal{O}_X$. $\square$.

Thm: (Riemann-Roch) let $f: X \to \mathrm{Spec}\, k$ be a proj. curve ω_X° be the dualizing sheaf for X, then $D \in \mathrm{Div}(X)$,

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \omega_f \otimes \mathcal{O}_X(-D)) = \deg D + 1 - p_a(X).$$

Pf: Existence of dualizing sheafs.

Def^n: X proper scheme of $\dim n / k$

A dualizing sheaf for X is a coherent sheaf ω_X° on X together with a trace morphism

$$\mathrm{tr}: H^n(X, \omega_X^\circ) \longrightarrow k \quad \text{s.t.}$$

$\forall \mathcal{F} \in \mathrm{Coh}(X)$, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}} k$$

gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \simeq H^n(X, \mathcal{F})^\vee$$

gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^0) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee.$$

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