Lecture 8: Proper and Affine Curves

Wednesday, March 18, 2020

Thm: Every one-dim'l proper scheme X over a field k is projective.
(noetherian)

Pf: Reduce to X integral + "non-singular"

(i) X - We have an exact sequence (Reduce to the case X is reduced)

$$0 \to \mathcal{I} \to \mathcal{O}_X \longrightarrow \mathcal{O}_{X_{red}} \to 0$$

ideal sheaf of nilpotents

& "factorize this sequence via sq. zero ideals"

If we have a square zero ideal sheaf, $\mathcal{I}^2 = 0$.
- $(\mathcal{I}/\mathcal{I}^2)$ k-module structure

Then we can actually restrict the above exact sequence

$$\boxed{0 \to \mathcal{I} \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{O}_{X_{red}}^* \longrightarrow 0}$$ (of groups)

$$a \longmapsto 1+a$$

X is noetherian $\Rightarrow \mathcal{I}^n = 0$ for some $n > 0$.

long exact sequence on cohomology

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X_{red}, \mathcal{O}_{X_{red}}^*) \to H^2(X, \mathcal{I}) = 0$$

|| surjective map ||

$Pic(X)$ $\quad$ $Pic(X_{red})$

Ex: $Pic(X) = H^1(X, \mathcal{O}_X^*)$

Then note that $\mathcal{L}$ is ample on X iff $\mathcal{L}_{red} = \mathcal{L} \otimes \mathcal{O}_{X_{red}}$ is ample on X_{red}.

$\mathcal{L}$ is ample on X i.e. $H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \ \forall n \geq n_0(\mathcal{F})$ (any coherent sheaf)

$$H^1(X_{red}, \mathcal{F} \otimes \mathcal{O}_{X_{red}} \otimes \mathcal{L}^n) = 0$$

$$0 \longrightarrow \mathcal{F}\mathcal{I} \otimes \mathcal{L}^n \to \mathcal{F}\otimes\mathcal{L}^n \to \mathcal{F} \otimes \mathcal{O}_{X_{red}} \otimes \mathcal{L}^n \to 0$$

$$\cancel{H^1(X, \mathcal{F}\otimes\mathcal{L}^n)} \to H^1(X_{red}, \mathcal{F}\otimes\mathcal{L}\otimes\mathcal{O}_{X_{red}}) \to 0 \leftarrow H^2(X, \mathcal{I}\mathcal{F}\otimes\mathcal{L}^n)$$

If X is projective $\Rightarrow X_{red}$ is proj.

X_{red} is proj $\Rightarrow X$ is projective

Step 2: Assume X is reduced but not necessarily irreducible
let $X_1, \ldots, X_n$ be irreducible components of X

Claim: $Pic(X) \longrightarrow \bigoplus Pic(X_i)$ is surjective

$\quad H^1(X, \mathcal{O}_X^*) \qquad \bigoplus H^1(X_i, \mathcal{O}_{X_i}^*)$

Hint: Use induction on n

$n = 2$ $\quad Pic(X) \to Pic(X_1) \oplus Pic(X_2)$

Meyer-Vietoris exact sequence for coh.

$X = X_1 \cup X_2, \quad X_1 \cap X_2 = pts.$

$\mathcal{L}$ is ample on X iff $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i for each i

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- $\mathcal{L}$ ample on X $\Rightarrow$ $\mathcal{L} \otimes \mathcal{O}_{X_i}$ ample as $X_i \hookrightarrow X$ closed immersion

$\Leftarrow$ Use induction $\mathcal{L}|_{X_1}$, $\mathcal{L}|_{X_2}$ are ample

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_{X_1} \to 0 \quad \text{exact}$$

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{X_2} \to 0$$

$H^1(X, (\mathcal{I}\mathcal{J}) \otimes \mathcal{L}^{\otimes n}) \to H^1(X, \mathcal{J} \otimes \mathcal{L}^n) \to 0$

support in X_1. $\mathcal{L}|_{X_1}$ ample $\to H^1(X, (\mathcal{J} \otimes \mathcal{L}^{\otimes n}) \otimes \mathcal{O}_{X_2})$

$\Rightarrow$ X is proj iff each X_i is projective. $\mathcal{L}|_{X_2}$ is ample.

can Assume X is integral.

We are left to show every proper integral curve is proj.

Reduction to X being non-singular.

normal $\Rightarrow$ non-singular in dim 1

$f : \tilde{X} \longrightarrow X$ be the normalization of X

if $\tilde{X}$ is projective $\Rightarrow$ X is projective.

$\mathcal{L}$ very ample invertible sheaf on $\tilde{X}$

$\Rightarrow$ $\exists$ $i : \tilde{X} \hookrightarrow \mathbb{P}^m$ for some m such that $\mathcal{L} = i^*\mathcal{O}(1)$

since the preimage in $\tilde{X}$ of singular points in X is finite pts, Bertini's thm, there exists a hyperplane $H \subset \mathbb{P}^m$ such that $D = i^*H = \sum P_i$ is an effective divisor on $\tilde{X}$ (EX: $\mathcal{O}_X(-D) = \mathcal{L}$)

and $f(P_i)$ are all non-singular pts. on X

$D_0 = \sum f(P_i)$, $\mathcal{L}_0 = \mathcal{O}_X(-D_0)$

$$\mathcal{L} = f^*\mathcal{L}_0.$$

Normalization morphism is finite.

EX: $\mathcal{L}_0$ is ample iff $f^*\mathcal{L}_0$ is ample.

"Use Cohomological Criterion" for ampleness

Now we just have to show that a non-singular integral proper curve is projective.

Ref^n: Hartshorne II 6.7, I 6.9 $\quad$ K: function field of tr deg 1 $/k$

C_K Abstract non-singular curve

set $\{$ all discrete valuation rings of K/k $\}$

Thm: Let X be a sep. curve over a field k.
Suppose that no irreducible component of X is proper
Then X is affine.

Lemma 1: Let X be an integral sep. scheme. Let $U \subset X$ be a non-empty affine open such that $X \setminus U$ is a finite set with $\mathcal{O}_{X,x_i}$ noetherian of dim 1.
Then $\exists$ a globally generated invertible sheaf $\mathcal{L}$ on X and a section s such that $U = X_s$.

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a section s such that $U = X_s$.

Proof: Say $U = \mathrm{Spec}(A)$, $K = k(X)$
$B_i = \mathcal{O}_{X,x_i}$ and $m_i = m_{x_i}$

$x_i \notin U$ $\quad U \times_X \mathrm{Spec}(B_i)$ has only one point!
$$\begin{array}{c} U \times_X \mathrm{Spec}(B_i) \\ \parallel \\ \mathrm{Spec}(K) \end{array}$$

X sep $\Rightarrow$ $\mathrm{Spec}(K)$ is a closed subscheme of $U \times_Z \mathrm{Spec}(B_i)$

(Ex.)
$\Rightarrow$ we can find a non-zero $f \in A$ such that $f^{-1} \in m_i$ $\forall$ $i = 1, \dots, r$.

$$\begin{tikzcd} U \times_X \mathrm{Spec}(B_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_Z \mathrm{Spec}(B_i) \arrow[d] \\ X \arrow[r, "\Delta_{X/Z}"'] & X \times X \end{tikzcd}$$

Pick open $U_i \subset X$, $x_i \in U_i$
s.t $f^{-1} \in \mathcal{O}(U_i)$

Then $\mathcal{U} : X = U \cup U_1 \dots \cup U_r$ is an open covering of X.

Consider the 2-cocycles with values in $\mathcal{O}_X^*$ given by
f on $U \cap U_i$ and 1 on $U_i \cap U_j$

This defines a line bundle $\mathcal{L}$ with two section
(1) s := 1 on U and f^{-1} on U_i
(2) section t := f on U and 1 on U_i.

$X_t \supset U_1 \cup \dots \cup U_r$.

Hence s, t generate $\mathcal{L}$. $\square$

<u>Lemma 2</u>: Let X be a quasi-compact scheme. If for every $x \in X$ there exists a pair $(\mathcal{L}, s)$ consisting of a globally generated invertible sheaf $\mathcal{L}$ and a global section s such that $x \in X_s$ and X_s is affine, then X has an ample invertible sheaf.

<u>Lemma 3</u> Let X be a Noetherian integral sep. scheme of dim 1. Then X has an ample invertible sheaf.

<u>Pf</u>: $X = U_1 \cup \dots \cup U_n$ affine open cover.
X noeth $\Rightarrow$ $X \setminus U_i$ is finite

Lemma 1 $\Rightarrow$ we can find a pair $(\mathcal{L}_i, s_i)$ consisting of globally generated sheaf $\mathcal{L}_i$ and global section s_i s.t $U_i = X_{s_i}$.

Lemma 2 $\Rightarrow$ X has an ample invertible sheaf. $\square$

<u>Prop^n</u>: X be an integral sep. curve $/k$. Then X is an affine scheme or X is projective.

<u>Pf</u>: Assume X is not projective.
Let $X \hookrightarrow \overline{X}$ be a open immersion to a projective scheme.
(X is quasi-projective + take closure of image)

Lemma 1 $\Rightarrow$ find a globally gen. invertible sheaf $\mathcal{L}$ on $\bar{X}$ and a section $s \in \Gamma(\bar{X}, \mathcal{L})$ such that $X = \bar{X}_s$.

Choose a basis $s = s_0, \ldots, s_m$ of the finite dim'l k-vectorspace $\Gamma(\bar{X}, \mathcal{L})$

$\Rightarrow f : \bar{X} \longrightarrow \mathbb{P}^m_k$ such that the inverse image of $D_+(T_0)$ is X. (EX.)

$f^{-1}(D_+(T_0)) = X$.

In particular, f is not-constant $\Rightarrow$ f maps generic pt η of $\bar{X}$ to a non-closed pt of $\mathbb{P}^n_k$

If $y \in \mathbb{P}^n_k$ is a closed pt., then $f^{-1}(\{y\})$ is a closed set of $\bar{X}$ not containing η, hence finite.

$\Rightarrow$ f is finite. (Ex)

$\Rightarrow$ $X = f^{-1}(D_+(T_0))$ is affine $\square$

Corollary: Let X be a sep. scheme of finite type over k dim $= 1$ and no irreducible comp. of X is proper $\Rightarrow X$ is affine.

Pf: X_i be irreducible components.

Apply Serre's Criterion for affineness

and show that

X_{red} is affine $\Leftrightarrow$ X is affine

$X_{(red)}$ is affine $\Leftrightarrow$ each X_i (red) is affine.