Lecture 10: Duality Theorem(s)

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<u>Not^n</u>: $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules $(X, \mathcal{O}_X)$-ringed space

$\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) :=$ group of $\mathcal{O}_X$-module hom.

$\mathcal{H}om_X(\mathcal{F}, \mathcal{G}) :=$ Hom sheaf $U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

For a fixed $\mathcal{F}$, $\mathrm{Hom}(\mathcal{F}, \cdot)$ left exact covariant functor $\mathrm{Mod}(X) \longrightarrow \mathrm{Ab}$

$\mathcal{H}om(\mathcal{F}, \cdot)$ ———— $\mathrm{Mod}(X) \to \mathrm{Mod}(X)$

Recall: $\mathrm{Mod}(X)$ has enough injectives

$\mathrm{Ext}^i(\mathcal{F}, \cdot)$ right derived functor of $\mathrm{Hom}(\mathcal{F}, \cdot)$

$\mathcal{E}xt^i(\mathcal{F}, \cdot)$ ———— of $\mathcal{H}om(\mathcal{F}, \cdot)$

- $\mathrm{Ext}^0 = \mathrm{Hom}$ $\qquad$ $\mathcal{E}xt^0 = \mathcal{H}om$
- $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > 0$, $\mathcal{G}$ injective $\qquad$ $\cdot$ $\mathcal{E}xt$

- $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$, $U \subseteq X$ open
- $\mathcal{G} \in \mathrm{Mod}(X)$:
  - $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$
  - $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0 \quad \forall\, i > 0$
  - $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G}) \quad i \geq 0$
- Short exact sequence $\longrightarrow$ long exact seq.
- $\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G})$

  $\mathcal{L}$ locally free sheaf of finite rank

  $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. dual.

- $\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee$.
- X noetherian scheme, $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{G} \in \mathrm{Mod}(X)$ $x \in X$

  $$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x), \; i \geq 0$$

- X proj scheme $/A$ noetherian ring

  $\mathcal{O}_X(1)$ very ample, $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$

Then $\exists$ an integer $n_0 > 0$ depending on $\mathcal{F}, \mathcal{G}$ and i, s.t

Then $\exists$ an integer $n_0 > 0$ depending on $\mathcal{F}, \mathcal{G}$ and i, s.t for every $n \geq n_0$, we have

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Thm: Let $X = \mathbb{P}^n_k$. $\underset{\mathcal{O}_X(-n-1)}{\overset{\parallel}{\omega_X}} = \Lambda^n \Omega_{X/k}$. Then

(a) $H^n(X, \omega_X) \cong k$. Fix one such isomorphism

(b) $\mathcal{F} \in \mathrm{Coh}(X)$, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \cong k$$

is a perfect pairing of finite dim'l vector spaces$/k$.

(c) for every $i \geq 0$, there is a natural functorial isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee$$

which for $i = 0$ is the one induced by the pairing b).

Proof: a) Thm 5.1. ($H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong k$)

b) $\mathcal{F} \to \omega_X$

induce $H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$

This gives the natural pairing.

If $\mathcal{F} = \mathcal{O}(q)$, then $\mathrm{Hom}(\mathcal{F}, \omega) \cong H^0(X, \omega(-q))$ and b) perfect pairing. Thm 5.1 d).

$\Rightarrow \mathcal{F} = \bigoplus_i \mathcal{O}(q_i)$

For arbitrary $\mathcal{F}$ $\quad \underset{\bigoplus_j \mathcal{O}(q_j)}{\mathcal{E}_1} \to \underset{\oplus^n \mathcal{O}(q_i)}{\mathcal{E}_0} \to \mathcal{F}_0 \to 0$

Apply 5-lemma.

(c) Exercise.

Def^n: X proper scheme of dim n/k

Dualizing sheaf: coherent sheaf ω_X° on X
+ $\mathrm{tr}: H^n(X, \omega_X^\circ) \to k$ s.t

$\forall \mathcal{F} \in \mathrm{Coh}(X)$ $\quad \mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}} k$

gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee.$$

Prop^n: X/k proper. Then a dualizing sheaf for X, if it exists, is unique. More precisely, $(\omega_X^\circ, \mathrm{tr})$ $(\omega_X', \mathrm{tr}')$ then there is a unique iso $\varphi: \omega_X^\circ \xrightarrow{\sim} \omega_X'$ s.t

$tr = tr' \circ H^n(\varphi)$.

Proof: ω' dualizing $\Rightarrow Hom(\omega^\circ, \omega') \cong H^n(X, \omega^\circ)^\vee$

$\Rightarrow \exists!$ morphism $\varphi : \omega^\circ \to \omega'$ corresponding to $tr \in H(X, \omega^\circ)^\vee$.

s.t $tr' \circ H^n(\varphi) = tr$.

ω° dualizing $\Rightarrow \exists! \; \psi : \omega' \to \omega^\circ$ st $tr \circ H^n(\psi) = tr'$

$\Rightarrow tr \circ H^n(\psi \circ \varphi) = tr$

But again ω° dualizing $\Rightarrow \psi \circ \varphi =$ identity map of ω°.

similarly $\varphi \circ \psi$ is identity. so φ is an iso.

Existence of dualizing sheaf for projective scheme $/k$

Lemma 1 Let X be a closed subscheme of codim r of $\mathbb{P}^N_k$. Then
$\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = 0$ for all $i < r$.

Proof: For any i, the sheaf $\mathcal{F}^i := \mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ is a coherent sheaf on $\mathbb{P}^N$.

($\mathcal{G}, \mathcal{F} \in Coh(X)$, then $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})$ is coherent sheaf)

$\mathcal{O}_{\mathbb{P}^N}(1)$ is a very ample line bundle

$\Rightarrow \quad \mathcal{F}^i \otimes \mathcal{O}(q) \quad q \gg 0$

is generated by global sections.

To show $\mathcal{F}^i$ is zero, it is sufficient to show that
$\Gamma(\mathbb{P}^N, \mathcal{F}^i(q)) = 0$ for all $q \gg 0$.

But $\quad \Gamma(\mathbb{P}^N, \mathcal{F}^i(q)) \cong Ext^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}(q)) \quad q \gg 0$

$$\cong H^{N-i}(\mathbb{P}^N, \mathcal{O}_X(-q))^\vee$$

For $i < r$ ($N - i > \dim X$) $\quad 0$ (Grothendieck's Vanishing).

$\square$

Lemma 2: $X \hookrightarrow \mathbb{P}^N_k$ codim r, let $\omega_X^\circ := \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$

Then for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a functorial iso

$$\mathcal{H}om_X(\mathcal{F}, \omega_X^\circ) \cong \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N})$$

Proof: let $0 \to \omega_{\mathbb{P}^N} \to \mathcal{I}^\circ$ an injective resolution of $\omega_{\mathbb{P}^N}$ in $Mod(\mathbb{P}^N)$.

$$\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}) = h^i(\mathcal{H}om_{\mathbb{P}^N}(\mathcal{F}, \mathcal{I}^\bullet))$$

Since $\mathcal{F}$ is an $\mathcal{O}_X$-module, any morphism $\mathcal{F} \to \mathcal{I}^i$ factors $\underset{\mathcal{O}_X\text{-module}}{\to} \mathcal{J}^i = \mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \mathcal{I}^i)$. (Exercise) $\mathcal{F} \to \mathcal{J}^i \to \mathcal{I}^i$

Thus, $\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}) = h^i(\mathcal{H}om_X(\mathcal{F}, \mathcal{J}^\bullet))$

Claim: $\mathcal{J}^i$ is injective $\mathcal{O}_X$-module.

[Indeed, $\mathcal{H}om_X(\mathcal{F}, \mathcal{J}^i) = \mathcal{H}om_{\mathbb{P}^N}(\mathcal{F}, \mathcal{I}^i)$

so $\mathcal{H}om_X(-, \mathcal{J}^i)$ is an exact functor.]

Furthermore, by lemma 1, we have $h^i(\mathcal{J}^\bullet) = 0$ for $i < r$.

So the complex is exact up to the r-th step.

$\Rightarrow$ $\mathcal{J}^\circ = \mathcal{J}_1 + \mathcal{J}_2$ where $\mathcal{J}_1$ is a complex in degree $0 \leq i \leq r$ and exact complex

$\mathcal{J}_2^\circ$ is a complex in degree $i \geq 2$

$$\mathcal{J}_1^{r-1} \xrightarrow{d^{r-1}} \underbrace{\mathcal{J}_1^r \oplus \mathcal{J}_2^r \xrightarrow{d^r} \mathcal{J}_2^{r+1}}$$

$$\omega_X^\circ = \frac{\ker d^r}{\operatorname{Im} d^{r-1}} = \frac{\cancel{\mathcal{J}_1^r} \oplus \ker(\ )}{\cancel{\mathcal{J}_1^r}}$$

$$= \ker(\ )$$

$$= \ker(\underline{\ \ })$$

$\Rightarrow \omega_X^\circ = \ker(d^r : \mathcal{J}_2^r \to \mathcal{J}_2^{r+1})$

and for any $\mathcal{O}_X$- module $\mathcal{F}$

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}).$$

Remark: $\left(\mathrm{Ext}^i_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}) = 0 \text{ for } i < r.\right)$ $\square$

Prop^n: let X be a projective scheme $/k$. Then X has a dualizing sheaf

Proof: let $X \hookrightarrow \mathbb{P}^N_k$ for some N, r codim

$$\omega_X^\circ = \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}).$$

Then lemma 2 $\Rightarrow$ for any $\mathcal{O}_X$- module $\mathcal{F}$,

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) = \mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}).$$

But the duality thm for $\mathbb{P}^N$, $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^N)$

we have an iso $\mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}) \cong H^{N-r}(\mathbb{P}^N, \mathcal{F})^\vee$.

But $N - r = n$, the dim X, $\mathcal{F}$ is a sheaf on X.

so we obtain a functorial iso $\mathcal{F} \in \mathrm{Coh}(X)$.

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

In particular take $\mathcal{F} = \omega_X^\circ$

$\mathrm{id} \in \mathrm{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ give a morphism

$id \in Hom_X(\omega_X^\circ, \omega_X^\circ)$ give a morphism of $H^n(X, \omega_X^\circ) \longrightarrow k$. $]\leftarrow$ Trace morphism.

Then (ω_X°, tr) is a dualizing sheaf for X. $\square$

Thm (Serre duality): X be a proj scheme of dim n/k let ω_X° be the dualizing sheaf on X and let $\mathcal{O}(1)$ be a very ample sheaf on X. Then:

(a) for all $i \geq 0$ and $\mathcal{F}$ coherent on X, there are natural functorial maps

$$\theta^i : Ext^i(\mathcal{F}, \omega_X^\circ) \longrightarrow H^{n-i}(X, \mathcal{F})^\vee,$$

such that θ° is the map given in the defn of dualizing sheaf.

b) TFAE:

(i) X is Cohen-Macaulay and equidimensional.
(dim X = depth X) for every local ring

(ii) for any $\mathcal{F}$ locally free on X, we have $H^i(X, \mathcal{F}(q)) = 0$ for $i < n$ and $q \gg 0$

(iii) the maps θ^i of (a) are isomorphisms for all $i \geq 0$ and all $\mathcal{F}$ coherent on X.

(Proof: Reading Assignment Hart III. Thm 7.6)

Thm: Let $X \hookrightarrow \mathbb{P}_k^N$ which is a local complete intersection of codim r. Then $\mathcal{I}$ to be the ideal sheaf of X in $\mathbb{P}_k^N$. Then $\omega_X^\circ \cong \omega_{\mathbb{P}^N} \otimes \Lambda^r(\mathcal{I}/\mathcal{I}^2)^\vee$

In particular, ω_X° is an invertible sheaf on X.

In particular. X is non-singular.

$\omega_X^\circ \cong \omega_X$. (Canonical sheaf)

Corollary 1: X projective non-singular curve.

Arithmetic genus = Geometric genus.

Corollary 2: X be a non-singular projective variety of dim n

$$H^q(X, \Omega^p) \cong H^q(X, \Omega^p)^\vee$$

$p+q = n \Rightarrow h^{p,q} = h^{q,p}$. $\square$.